

A q -Analog of Transposition Symmetry for Invariant G -Functions

R. A. GUSTAFSON AND S. C. MILNE*

*Department of Mathematics, Texas A & M University,
College Station, Texas 77843*

Submitted by G.-C. Rota

We give an algebraic construction of Milne's ${}_{\mu}^v[G]_m^{(n)}(\gamma; \delta; z)$ functions in terms of the matrix coefficients of certain linear operators $B_{\mu; z}$. We show that the ${}_{\mu}^v[G]_m^{(n)}(\gamma; \delta; z)$ functions all satisfy "transposition" symmetry if and only if $z = (t, \dots, t) \in \mathbb{C}^n$, unless ${}_{\mu}^v[G]_1^{(n)}(\gamma; \delta; z)$ is identically zero. This transposition symmetry is a " q -analog" of that for the ordinary ${}_{\mu}^v G_m^{(n)}(\gamma; \delta)$ polynomials of Biedenharn, Gustafson, and Milne. We also define a " q -analog" of the elementary reduced Wigner coefficients for $U(n)$ and give both a combinatorial and representation theoretic interpretation of this definition. © 1986 Academic Press, Inc.

0. INTRODUCTION

In [17] Milne introduced a family of q -analogues ${}_{\mu}^v[G]_m^{(n)}(\gamma_1, \dots, \gamma_n; \delta_1, \dots, \delta_n | z_1, \dots, z_n) \equiv {}_{\mu}^v[G]_m^{(n)}(\gamma; \delta | (z))$ of the $U(n)$ invariant special functions ${}_{\mu}^m G_q^{(n)}(\gamma_1, \dots, \gamma_n; \delta_1, \dots, \delta_n)$ which arise in the explicit computation of the "stretched" Wigner coefficients for the unitary groups $U(n)$. For special cases of the ${}_{\mu}^v[G]_m^{(n)}$ functions, Milne proved in [16, 17] a kind of "transposition" symmetry, which is a q -analog of the "ordinary" transposition symmetry discussed in [13]. The proof of these special cases of "transposition" symmetry in [16, 17] is direct and highly computational whereas the proof in the ordinary case [13] is a consequence of the limiting properties of the Biedenharn–Louck pattern calculus and the Weyl algebra of creation and annihilation operators. The elegant, though formidable, machinery involved in the proof of transposition symmetry for the ordinary ${}_{\mu}^m G_q^{(n)}$ functions is the subject of the lengthy paper [9].

In the present paper we give first a natural algebraic construction for the ${}_{\mu}^v[G]_m^{(n)}$ functions, and second, we show that they satisfy a general "transposition" symmetry analogous to that of the ${}_{\mu}^m G_q^{(n)}$ functions. As it

* Partially supported by NSF Grant MCS-8301647.

turns out, it is not necessary to reconstruct the whole machinery of the boson calculus as in [9]. It suffices to define a family of operators $B_{\mu; z}$, where μ is a nonnegative integer and $z \in \mathbb{C}^n$, and compute the matrix coefficients of products of these operators. The $B_{\mu; z}$ operators can be viewed as q -analogs of the denominator factors in the Biedenharn–Louck pattern calculus [7]. In sections 1 and 2 we show that the commutativity “in general position” of the $B_{\mu; z}$ and $B_{0; z}$ operators is equivalent to the transposition symmetry of the ${}_{\mu}[G]_m^{(n)}(\gamma_1, \dots, \gamma_n; \delta_1, \dots, \delta_n; z_1, \dots, z_n)$ functions (which are equivalent to the ${}_{\mu}[G]_m^{(n)}(\gamma; \delta | (z))$ functions). After computing ${}_{\mu}[G]_1^{(n)}(\gamma; \delta; z)$ as a sum of products of Schur functions we then show directly that ${}_{\mu}[G]_1^{(n)}(\gamma; \delta; z)$ (if not identically zero) satisfies transposition symmetry if and only if $z = (t, \dots, t) \in \mathbb{C}^n$, where $t \in \mathbb{C}$. It then follows that all the ${}_{\mu}[G]_m^{(n)}(\gamma; \delta; z)$ functions satisfy transposition symmetry whenever $z = (t, \dots, t) \in \mathbb{C}^n$.

In Section 3 we show that the general ${}_{\mu}[G]_m^{(n)}$ functions which are q -analogs of the ${}_{\mu}G_q^{(n)}$ functions are limiting values of the ${}_{\mu}[G]_m^{(n)}$ functions. The transposition symmetry for the general ${}_{\mu}[G]_m^{(n)}$ functions is then a consequence of that for the ${}_{\mu}[G]_m^{(n)}$ functions.

In Section 4 we give a brief description of the changes in the pattern calculus [7] needed to produce q -analogs of the $U(n)$ Wigner coefficients and, via the Biedenharn path sum formula (see [9]), the q -analog of the ${}_{\mu}G_q^{(n)}$ function. In [10], Biedenharn and Ciftan discuss a beautiful algebraic and combinatorial description of the “elementary reduced” Wigner coefficients (the building blocks of the Racah–Wigner algebra) in terms of the degrees of irreducible representations of the symmetric groups or in terms of combinatorial hook length formulas for standard tableaux. To obtain the q -analog one replaces the degrees of the irreducible representations of the symmetric group \mathcal{S}_N by the degrees of corresponding irreducible representation of $Gl_N(F_q)$ which occur in inducing the trivial character from the upper triangular subgroup of $Gl_N(F_q)$. From a combinatorial point of view, the hook length formulas are replaced by q -hook length formulas, which are generating functions for “ q -counting” standard Young tableaux (see pp. 50–53 of [14]).

A topic that is not considered here is the development of a q -analog of the Racah–Wigner algebra. We have implicitly defined such a q -analog in Definition 4.1. There is, however, much to be done in extending the important properties of the classical Racah–Wigner algebra to this q -analog. The development of these properties is the subject of [11].

We also mention that the special case ${}^{j-2}_0 G_q^2$ for $j \geq 2$ is up to a simple factor the well-poised hypergeometric series ${}_{j+1}F_j$. In [13, Theorem 1.19] the transposition symmetry of the polynomial ${}^{j-2}_0 G_q^2$ implies an interesting new contiguous relation for the well-poised hypergeometric series ${}_{j+1}F_j$. In [16] Milne gives a direct proof of the analogous contiguous relation for

the well-poised *basic* hypergeometric series ${}_j+2\phi_{j+1}$. The transposition symmetry of Proposition 3.15 provides a natural generalization of these new contiguous relations for well-poised basic hypergeometric series.

1. $B_{\mu; z}$ OPERATORS, MATRIX COEFFICIENTS AND A SUMMATION THEOREM

We consider the set A_n of all n -part partitions where n is a positive integer. An n -part partition $\lambda = (\lambda_1, \dots, \lambda_n)$ is an n -tuple of non-negative integers such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$. Define ρ to be the partition $(n-1, \dots, 0)$ whose i th part is $n-i$ and $p = (p_1, \dots, p_n)$ to be the vector sum $\lambda + p = (\lambda_1 + n-1, \dots, \lambda_n + 0)$. If S is a subset of $I_n = \{1, \dots, n\}$ then let $A(S)$ be the n -tuple $(\chi(1 \in S), \chi(2 \in S), \dots, \chi(n \in S))$, where $\chi(T)$ is 1 if statement T is true and 0 if T is false.

Let \mathcal{V}_n be the infinite dimensional Hilbert space over the complex numbers with hermitian inner product (\cdot, \cdot) and orthonormal basis $\{\Phi_\lambda | \lambda \in A_n\}$.

DEFINITION 1.1. For any nonnegative integer μ such that $0 \leq \mu \leq n-1$, any $z = (z_1, \dots, z_n) \in \mathbb{C}$, and any positive real number q such that $q \neq 1$, we define $B_{\mu; z}$ to be the bounded linear operator on \mathcal{V}_n ,

$$B_{\mu; z}(\Phi_\lambda) = \sum_{\substack{S \subseteq I_n \\ \|S\| = \mu+1}} \prod_{k \in S^c} z_k \left[\prod_{\substack{i < j \\ i \in S, j \in S^c}} \frac{(1-q)}{(1-q^{p_i-p_j})} \prod_{\substack{i < j \\ i \in S^c, j \in S}} \frac{(1-q)}{(q^{p_j-p_i}-1)} \Phi_{\lambda+A(S)} \right], \quad (1.2)$$

where $\lambda \in A_n$, $S^c = I_n - S$, and the sum is over all subsets $S \subseteq I_n$ of cardinality $\mu+1$ such that $\lambda + A(S) \in A_n$. In the special case when $z = (1, \dots, 1)$ we shall denote $B_{\mu; z}$ by simply B_μ .

DEFINITION 1.3. Let $W = \mathbb{Z}^+ \cup \{0\}$, the set of all nonnegative integers. Also let W^n denote the set of all n -tuples of nonnegative integers.

We are interested in computing the matrix coefficients of products of the $B_{\mu; z}$ operators.

DEFINITION 1.4. Let $l \geq m \geq 0$ be integers and $[A] = [A_1, \dots, A_n] \in \mathbb{Z}^n$ satisfy $\sum_{i=1}^n A_i = l + \mu m$. Let $\lambda \in \mathbb{Z}^n$ and set $p = \lambda + \rho$, where ρ is defined above.

If $\lambda, \lambda + [A] \in A_n$ and $[A] \in W^n$, then define

$${}_\mu D_m^{(n)}([A]; p; z) = (\Phi_{\lambda+[A]}, B_{\mu; z}^m B_0^{l-m} \Phi_\lambda) \quad (1.5a)$$

and

$${}_{\mu}\tilde{D}_m^{(n)}([\Delta]; p; z) = (\Phi_{\lambda + [\Delta]}, B_{0;\mu;z}^{l-m} B_{\mu;z}^m \Phi_{\lambda}). \quad (1.5b)$$

If $\lambda \notin A_n$, $\lambda + [\Delta] \notin A_n$ or $[\Delta] \notin W^n$, then define

$${}_{\mu}D_m^{(n)}([\Delta]; p; z) \equiv {}_{\mu}\tilde{D}_m^{(n)}([\Delta]; p; z) \equiv 0. \quad (1.6)$$

In the special case when $z = (1, \dots, 1)$ and with assumptions as in (1.5a and b) or (1.6), we set

$${}_{\mu}D_m^{(n)}([\Delta]; p) \equiv {}_{\mu}D_m^{(n)}([\Delta]; p; 1, \dots, 1) \quad (1.7)$$

and similarly define ${}_{\mu}\tilde{D}_m^{(n)}([\Delta]; p)$.

Remark 1.8. In the case $\mu = 0$, note that

$${}_0D_m^{(n)}([\Delta]; p) = {}_0\tilde{D}_m^{(n)}([\Delta]; p) = {}_0\tilde{D}_l^{(n)}([\Delta]; p), \quad (1.9)$$

where $\lambda, [\Delta] \in \mathbb{Z}^n$ and $[\Delta]$ satisfies $\sum_{i=1}^n \Delta_i = l$ with $l \geq m \geq 0$.

An immediate consequence of Definition 1.4 is that the commutativity of the $B_{\mu;z}$ and B_0 operators is equivalent to

$${}_{\mu}D_m^{(n)}([\Delta]; p; z) = {}_{\mu}\tilde{D}_m^{(n)}([\Delta]; p; z) \quad (1.10)$$

for all $l \geq m \geq 0$, all $[\Delta] \in W^n$ and all $\lambda \in A_n$ such that $\sum_{i=1}^n \Delta_i = l + \mu m$. We will prove a weaker result than (1.10) in this paper. Instead of the full commutativity of the $B_{\mu;z}$ and B_0 operators we will be considering the following

DEFINITION 1.11. We shall say that the operators $B_{\mu;z}$ and B_0 commute in general position if there exists a positive real number $c > 0$ such that, for all $\lambda \in A_n$ with

$$\lambda_1 > \lambda_2 + c > \lambda_3 + 2c > \dots > \lambda_n + (n-1)c, \quad (1.12a)$$

we have

$$B_{\mu;z} B_0(\Phi_{\lambda}) = B_0 B_{\mu;z}(\Phi_{\lambda}). \quad (1.12b)$$

The connection between the commutativity of the $B_{\mu;z}$ and B_0 operators and the equivalence of the ${}_{\mu}D_m^{(n)}([\Delta]; p; z)$ and ${}_{\mu}\tilde{D}_m^{(n)}([\Delta]; p; z)$ functions is given by

LEMMA 1.13. Let $n-1 \geq \mu \geq 0$ and $z \in \mathbb{C}^n$. The operators $B_{\mu;z}$ and B_0 commute in general position if and only if there exists a positive real number

$c > 0$ such that, for all $\lambda \in A_n$ and all $[\Delta] \in W^n$ satisfying $\sum_{i=1}^n \Delta_i = \mu + 2$ and $\lambda_1 > \lambda_2 + c > \cdots > \lambda_n + (n-1)c$, we have

$${}_{\mu}D_1^{(n)}([\Delta]; p; z) = {}_{\mu}\tilde{D}_1^{(n)}([\Delta]; p; z). \quad (1.14)$$

If $B_{\mu;z}$ and B_0 commute in general position then for every $l \geq m \geq 0$ there exists a positive real number $c(l, m) > 0$ such that, for $\lambda \in A_n$ and $[\Delta] \in W^n$ satisfying $\sum_{i=1}^n \Delta_i = l + \mu m$ and $\lambda_1 > \lambda_2 + c(l, m) > \cdots > \lambda_n + (n-1)c(l, m)$, we have

$${}_{\mu}D_m^{(n)}([\Delta]; p; z) = {}_{\mu}\tilde{D}_m^{(n)}([\Delta]; p; z). \quad (1.15)$$

Proof. It follows from Definitions 1.1, 1.4, and 1.11 that $B_{\mu;z}$ and B_0 commute in general position if and only if identity (1.14) is true.

Identity (1.15) is proved by induction. We assume inductively that $k \geq 0$ is an integer such that for every l, m with $k \geq l \geq m \geq 0$, there exists a positive real number $c(l, m) > 0$ so that identity (1.15) holds. If $m = 0$ there is nothing to prove, so we can assume that $m > 0$.

For simplicity let $c(k) > 0$ be chosen so that $c(k) \geq c(l, m)$ for all l, m such that $k \geq l \geq m \geq 0$ and $c(k) \geq c$, where c is defined as in Definition 1.11.

If $l = k + 1$ and $l > m > 0$, then by Definitions 1.1, 1.4 and the induction hypothesis we have

$$\begin{aligned} {}_{\mu}D_m^{(n)}([\Delta]; p; z) &= (\Phi_{\lambda + [\Delta]}, B_{\mu;z}^m B_0^{l-m} \Phi_{\lambda}) \\ &= (\Phi_{\lambda + [\Delta]}, B_{\mu;z}^m B_0^{l-1-m} (B_0 \Phi_{\lambda})) \\ &= (\Phi_{\lambda + [\Delta]}, B_0^{l-1-m} B_{\mu;z}^m (B_0 \Phi_{\lambda})) \end{aligned} \quad (1.16)$$

for all $[\Delta] \in W^n$ and $\lambda \in A_n$ satisfying $\sum_{i=1}^n \Delta_i = l + \mu m$ and $\lambda_1 > \lambda_2 + c(k) + 1 > \lambda_3 + 2c(k) + 2 > \cdots > \lambda_n + (n-1)c(k) + (n-1)$. Hence it follows from the assumed commutativity in general position of $B_{\mu;z}$ and B_0 that

$$\begin{aligned} {}_{\mu}D_m^{(n)}([\Delta]; p; z) &= (\Phi_{\lambda + [\Delta]}, B_0^{l-1-m} B_{\mu;z}^{m-1} (B_{\mu;z} B_0 \Phi_{\lambda})) \\ &= (\Phi_{\lambda + [\Delta]}, B_0^{l-1-m} B_{\mu;z}^{m-1} (B_0 B_{\mu;z} \Phi_{\lambda})) \end{aligned} \quad (1.17)$$

for all $[\Delta] \in W^n$ and $\lambda \in A_n$ satisfying $\sum_{i=1}^n \Delta_i = l + \mu m$ and $\lambda_1 > \lambda_2 + (2c(k) + 2) > \lambda_3 + 2(2c(k) + 2) > \cdots > \lambda_n + (n-1)(2c(k) + 2)$. Finally from Definition 1.1 and the induction hypothesis we obtain

$$\begin{aligned} {}_{\mu}D_m^{(n)}([\Delta]; p; z) &= (\Phi_{\lambda + [\Delta]}, B_0^{l-1-m} (B_{\mu;z}^{m-1} B_0) (B_{\mu;z} \Phi_{\lambda})) \\ &= (\Phi_{\lambda + [\Delta]}, B_0^{l-1-m} (B_0 B_{\mu;z}^{m-1}) (B_{\mu;z} \Phi_{\lambda})) \\ &= (\Phi_{\lambda + [\Delta]}, B_0^{l-m} B_{\mu;z}^m \Phi_{\lambda}) = {}_{\mu}\tilde{D}_m^{(n)}([\Delta]; p; z) \end{aligned} \quad (1.18)$$

for all $[\Delta] \in W^n$ and $\lambda \in A_n$ with $\sum_{i=1}^n \Delta_i = l + \mu m$ and $\lambda_1 > \lambda_2 + 3c(k) + 3 > \dots > \lambda_n + (n-1)3c(k) + (n-1)3$. This proves identity (1.15) in the case $l = k+1$ and $l > m > 0$.

The proof of identity (1.15) in the case $l = k+1$ and $l = m > 0$ is very similar to the above. Hence (1.15) follows by induction for all $k > 0$.

Q.E.D.

The multiplication of the $B_{\mu; z}$ operators implies general recursion relations for the ${}_{\mu}D_m^{(n)}([\Delta]; p; z)$ and ${}_{\mu}\tilde{D}_m^{(n)}([\Delta]; p; z)$ functions.

LEMMA 1.19. *Let $l \geq m \geq 1$ and $[\Delta] \in \mathbb{Z}^n$ satisfy $\sum_{i=1}^n \Delta_i = l + \mu m$. If $\lambda \in \mathbb{Z}^n$ and $p = \lambda + \rho$ then*

$$\begin{aligned} {}_{\mu}D_m^{(n)}([\Delta]; p; z) &= \sum_{\substack{S \subseteq I_n \\ \|S\| = \mu + 1}} {}_{\mu}D_1^{(n)}([\Delta(S)]; p + [\Delta] - [\Delta(S)]; z) \\ &\quad \times {}_{\mu}D_{m-1}^{(n)}([\Delta] - [\Delta(S)]; p; z) \end{aligned} \quad (1.20)$$

and

$$\begin{aligned} {}_{\mu}\tilde{D}_m^{(n)}([\Delta]; p; z) &= \sum_{\substack{S \subseteq I_n \\ \|S\| = \mu + 1}} {}_{\mu}\tilde{D}_{m-1}^{(n)}([\Delta] - [\Delta(S)]; p + [\Delta(S)]; z) \\ &\quad \times {}_{\mu}\tilde{D}_1^{(n)}([\Delta(S)]; p; z), \end{aligned} \quad (1.21)$$

where the sums in (1.20) and (1.21) are over all subsets $S \subseteq I_n$ of cardinality $\mu + 1$ and $[\Delta(S)]$ is defined at the beginning of this section.

Proof. We shall assume that $\lambda, \lambda + [\Delta] \in A_n$ and $[\Delta] \in W^n$, for otherwise both sides of (1.20) and (1.21) are identically zero. Also we give only the proof of (1.20) here. The proof of (1.21) is very similar.

First, observe that by Definition 1.4 and (1.6) we may restrict the sum in (1.20) to subsets $S \subseteq I_n$ of cardinality $\mu + 1$ such that $\lambda + [\Delta] - [\Delta(S)] \in A_n$ and $[\Delta] - [\Delta(S)] \in W^n$. With this restriction, Definitions 1.1 and 1.4 show that formula (1.20) is equivalent to the following:

$$\begin{aligned} &({}_{\mu}\Phi_{\lambda + [\Delta]}, B_{\mu; z}^m B_0^{l-m} \Phi_{\lambda}) \\ &= \sum_{\substack{S \subseteq I_n \\ \|S\| = \mu + 1}} ({}_{\mu}\Phi_{\lambda + [\Delta]}, B_{\mu; z} \Phi_{\lambda + [\Delta] - [\Delta(S)]}) \\ &\quad \times ({}_{\mu}\Phi_{\lambda + [\Delta] - [\Delta(S)]}, B_{\mu; z}^{m-1} B_0^{(l-1) - (m-1)} \Phi_{\lambda}), \end{aligned} \quad (1.22)$$

where the sum is over all subsets $S \subseteq I_n$ of cardinality $\mu + 1$ such that

$\lambda + [\Delta] - [\Delta(S)] \in A_n$ and $[\Delta] - [\Delta(S)] \in W^n$. Equation (1.22) is the usual expression for the matrix coefficients of the operator $B_{\mu;z}^m B_0^{l-m}$ in terms of the matrix coefficients of the operators $B_{\mu;z}^m$ and $B_{\mu;z}^{m-1} B_0^{l-m}$, i.e., matrix multiplication of the matrices of the operators $B_{\mu;z}^m$ and $B_{\mu;z}^{m-1} B_0^{l-m}$.
Q.E.D.

In Proposition 1.25 we shall prove a summation formula for a special case of (1.7). We begin by defining the q -factorial of a nonnegative integer.

DEFINITION 1.23. If k is a positive integer define

$$[k]! = (1-q)(1-q^2) \cdots (1-q^k)(1-q)^{-k}. \quad (1.24)$$

If $k=0$ define $[0]! = 1$.

The following proposition is a " q -analog" of formula (3.46a) of [3].

PROPOSITION 1.25. Let $l \geq 0$ be an integer and $[\Delta] \in W^n$ satisfy $\sum_{i=1}^n \Delta_i = l$. If $\lambda \in A_n$ and $p = \lambda + \rho$, then define $r = (r_1, \dots, r_n) = [\Delta] + p$ to be the n -tuple with $r_i = \Delta_i + \lambda_i + n - i$ in the i th position. We shall assume that $r_i \geq p_i > r_j + 1$ whenever $1 \leq i < j \leq n$. With notation as in (1.7) and (1.9) we have

$$\begin{aligned} {}_0D_l^{(n)}([\Delta]; p) &= q^{\sum_{i < j=1}^n \left[\binom{p_i - p_j + 1}{2} - \binom{p_i - r_j + 1}{2} \right]} \left((1-q)^{-\binom{n}{2}} [l]! \left/ \prod_{i=1}^n [\Delta_i]! \right. \right) \\ &\quad \times \prod_{i < j=1}^n \frac{[p_i - r_j - 1]!}{[r_i - p_j]!} (1 - q^{r_i - r_j}). \end{aligned} \quad (1.26)$$

Proof. As a special case of Lemma 1.19 we obtain the following recursion relation:

$$\begin{aligned} {}_0D_l^{(n)}([\Delta]; p) &= \sum_{\tau=1}^n {}_0D_1^{(n)}([\Delta(\tau)]; p + [\Delta] - [\Delta(\tau)]) {}_0D_{l-1}^{(n)}([\Delta] - [\Delta(\tau)]; p), \end{aligned} \quad (1.27)$$

where $[\Delta(\tau)]$ is the n -tuple with one in the τ th position and zero elsewhere.

Note that if $l \geq 1$ and $\Delta_\tau \geq 1$, where Δ_τ is the τ th component of $[\Delta]$, then ${}_0D_{l-1}^{(n)}([\Delta] - [\Delta(\tau)]; p)$ satisfies the assumptions of Proposition 1.25. Also note that since $p_i > r_j + 1$ for all integers i, j with $1 \leq i < j \leq n$ then $\lambda + [\Delta] - [\Delta(\tau)] \in A_n$ and ${}_0D_1^{(n)}([\Delta(\tau)]; p + [\Delta] - [\Delta(\tau)])$ is a positive real number from Definitions 1.1 and 1.4. Applying recursion relation (1.27) it follows that ${}_0D_l^{(n)}([\Delta]; p)$ is a positive real number.

We shall now prove (1.26) by showing that (1.26) satisfies the recursion

relation (1.27). First, we observe that ${}_0D_0^{(n)}([0, \dots, 0]; p) = 1$. From (1.2) we obtain, for $1 \leq \tau \leq n$,

$${}_0D_1^{(n)}([\mathcal{A}(\tau)]; p) = \prod_{\tau < j \leq n} \frac{(1-q)}{(1-q^{p_\tau - p_j})} \prod_{1 \leq i < \tau} \frac{(1-q)}{(q^{p_\tau - p_i} - 1)} \quad (1.28a)$$

$$= \prod_{1 \leq i < \tau} q^{p_i - p_\tau} \prod_{\tau < j \leq n} \frac{(1-q)}{(1-q^{p_\tau - p_j})} \prod_{1 \leq i < \tau} \frac{(1-q)}{(1-q^{p_i - p_\tau})}. \quad (1.28b)$$

Setting $r(\tau) = [r(\tau)_1, \dots, r(\tau)_n] = p + [\mathcal{A}(\tau)]$ we find

$$\begin{aligned} {}_0D_1^{(n)}([\mathcal{A}(\tau)]; p) &= q^{\sum_{i < j=1}^n [(\frac{p_i - p_j + 1}{2}) - (\frac{p_i - r(\tau)_j + 1}{2})]} \\ &\times \prod_{\tau < j \leq n} \frac{(1-q)}{(1-q^{p_\tau - p_j})} \prod_{1 \leq i < \tau} \frac{(1-q)}{(1-q^{p_i - p_\tau})}. \end{aligned} \quad (1.29)$$

Applying (1.29) one then verifies formula (1.26) for ${}_0D_1^{(n)}([\mathcal{A}(\tau)]; p)$.

We now complete the proof of (1.26) by induction on l . We assume that (1.26) is true for all $0 \leq l < k$ and verify it for $l = k$. Under the assumptions of Proposition 1.25 we find that ${}_0D_l([\mathcal{A}]; p)$ as given by (1.26) is not zero. We therefore rewrite relation (1.27) as

$$\begin{aligned} \sum_{\tau=1}^n \{ {}_0D_l^{(n)}([\mathcal{A}]; p) \}^{-1} {}_0D_1^{(n)}([\mathcal{A}(\tau)]; p + [\mathcal{A}] - [\mathcal{A}(\tau)]) \\ \times {}_0D_{l-1}^{(n)}([\mathcal{A}] - [\mathcal{A}(\tau)]; p) = 1. \end{aligned} \quad (1.30)$$

We are not assuming (1.30) is true, but need to prove that ${}_0D_1^{(n)}([\mathcal{A}]; p)$ as given by (1.26) satisfies (1.30). Substituting into (1.30) the expressions (1.26) and (1.29), we are reduced to verifying

$$\begin{aligned} \sum_{\tau=1}^n q^{\sum_{i < \tau} (r_i - p_i)} \frac{(1-q^{\mathcal{A}_\tau})}{(1-q^l)} \\ \times \prod_{\tau < j \leq n} \left[\frac{(1-q^{r_\tau - p_j})}{(1-q)} \frac{(1-q^{(r_\tau - r_j - 1)})}{(1-q^{r_\tau - r_j})} \frac{(1-q)}{(1-q^{(r_\tau - r_j - 1)})} \right] \\ \times \prod_{1 \leq i < \tau} \left[\frac{(1-q^{p_i - r_\tau})}{(1-q)} \frac{(1-q^{(r_i - r_\tau + 1)})}{(1-q^{r_i - r_\tau})} \frac{(1-q)}{(1-q^{(r_i - r_\tau + 1)})} \right] \\ = 1, \end{aligned} \quad (1.31)$$

where \mathcal{A}_τ is the τ th component of $[\mathcal{A}]$.

After simplification (1.31) becomes

$$\sum_{\tau=1}^n q^{\sum_{i<\tau} A_i} \frac{(1-q^{A_\tau})}{(1-q^l)} \prod_{\tau < j \leq n} \frac{(1-q^{r_\tau - p_j})}{(1-q^{r_\tau - r_j})} \times \prod_{1 \leq i < \tau} \frac{(1-q^{p_i - r_\tau})}{(1-q^{r_i - r_\tau})} = 1. \quad (1.32)$$

Substituting into (1.32) the expression

$$\frac{(1-q^{p_i - r_\tau})}{(1-q^{r_i - r_\tau})} = q^{-A_i} \frac{(1-q^{r_\tau - p_i})}{(1-q^{r_\tau - r_i})}, \quad (1.33)$$

where $1 \leq i < \tau$, and setting $q^{r_j} = \gamma_j$ and $q^{p_j} = \delta_j$ for $1 \leq j \leq n$, we rewrite (1.32) as

$$(1-q^l)^{-1} \sum_{\tau=1}^n \left[\prod_{j=1}^n (1 - (\gamma_\tau / \delta_j)) \prod_{\substack{j=1 \\ j \neq \tau}}^n (1 - (\gamma_\tau / \gamma_j))^{-1} \right] = 1. \quad (1.34)$$

We will finish the proof of expression (1.34) and Proposition 1.25 by observing that (1.34) is a special case of the following

LEMMA 1.35 (Lemma 1.33 of [17]). *Let $x_1, \dots, x_n, y_1, \dots, y_n$ be indeterminates where $y_i \neq y_j$ for $1 \leq i < j \leq n$, then*

$$1 - (x_1 \cdots x_n) = \sum_{p=1}^n (1 - x_p) \prod_{\substack{i=1 \\ i \neq p}}^n \frac{(y_p - x_i y_i)}{(y_p - y_i)}. \quad (1.36)$$

Setting $x_i = (\gamma_i / \delta_i) = q^{A_i}$ and $y_i = \gamma_i^{-1}$ for $1 \leq i \leq n$ into (1.36) we obtain

$$(1 - q^{\sum_{i=1}^n A_i}) = \sum_{\tau=1}^n \left[\prod_{j=1}^n (1 - (\gamma_\tau / \delta_j)) \prod_{\substack{j=1 \\ j \neq \tau}}^n (1 - (\gamma_\tau / \gamma_j))^{-1} \right], \quad (1.37)$$

which becomes identical to (1.34) after substituting $l = \sum_{j=1}^n A_j$ and dividing both sides of (1.37) by $(1 - q^l)$. This completes the proof of Proposition 1.25. Q.E.D.

2. ${}_\mu[G]_m^{(n)}$ FUNCTIONS

In this section we will define the ${}_\mu[G]_m^{(n)}$ and ${}_\mu[\tilde{G}]_m^{(n)}$ functions and give q -difference equations for them. We will also discuss their symmetries including "transposition" symmetry.

DEFINITION 2.1. As above let $p = \lambda + \rho$ where $\lambda \in A_n$ and $z =$

$(z_1, \dots, z_n) \in \mathbb{C}^n$. Let l, m, μ be integers such that $l \geq m \geq 0$ and $n-1 \geq \mu \geq 0$, and $[\Delta] = [\Delta_1, \dots, \Delta_n] \in \mathcal{W}^n$ satisfy $\sum_{i=1}^n \Delta_i = l + \mu m$. We further assume that $p_i > p_j + \Delta_j + 1$ whenever $1 \leq i < j \leq n$. Then define

$$\gamma_i = q^{p_i + \Delta_i} \quad \text{and} \quad \delta_i = q^{p_i}, \quad (2.2)$$

where $1 \leq i \leq n$, and define

$$\begin{aligned} {}_\mu[G]_m^{(n)}(\gamma; \delta; z) &= {}_\mu[G]_m^{(n)}(\gamma_1, \dots, \gamma_n; \delta_1, \dots, \delta_n; z_1, \dots, z_n) \\ &= \frac{[l + \mu m]!}{[l - m]!} \{ {}_0D_{l+\mu m}^{(n)}([\Delta]; p) \}^{-1} {}_\mu D_l^{(n)}([\Delta]; p; z) \end{aligned} \quad (2.3a)$$

and

$$\begin{aligned} {}_\mu[\tilde{G}]_m^{(n)}(\gamma; \delta; z) &= {}_\mu[\tilde{G}]_m^{(n)}(\gamma_1, \dots, \gamma_n; \delta_1, \dots, \delta_n; z_1, \dots, z_n) \\ &= \frac{[l + \mu m]!}{[l - m]!} \{ {}_0D_{l+\mu m}^{(n)}([\Delta]; p) \}^{-1} {}_\mu \tilde{D}_l^{(n)}([\Delta]; p; z). \end{aligned} \quad (2.3b)$$

Lemma 1.19 and Proposition 1.25 imply the following q -difference equations for ${}_\mu[G]_m^{(n)}(\gamma; \delta; z)$ and ${}_\mu[\tilde{G}]_m^{(n)}(\gamma; \delta; z)$.

PROPOSITION 2.4. *With assumptions and notation as in Definition 2.1 we have*

$$\begin{aligned} {}_\mu[G]_m^{(n)}(\gamma; \delta; z) &= (-1)^{\binom{\mu+1}{2}} q^{-\binom{\mu+1}{2}} \\ &\quad \times \sum_{\substack{\|S\| = \mu+1 \\ S \subseteq I_n}} \prod_{k \in S^c} z_k \prod_{\substack{i \in S \\ 1 \leq i \leq n}} \frac{(1 - \gamma_i / \delta_i)}{(1 - q)} \prod_{\substack{i \in S \\ j \in S^c}} \frac{(1 - q)}{(1 - \gamma_i / \gamma_j)} \\ &\quad \times {}_\mu[G]_{m-1}^{(n)}(\gamma_i q^{-\chi(i \in S)}; \delta; z) \end{aligned} \quad (2.5a)$$

and

$$\begin{aligned} {}_\mu[\tilde{G}]_m^{(n)}(\gamma; \delta; z) &= (-1)^{\binom{\mu+1}{2}} q^{-\binom{\mu+1}{2}} \\ &\quad \times \sum_{\substack{\|S\| = \mu+1 \\ S \subseteq I_n}} \prod_{k \in S^c} z_k \prod_{\substack{i \in S \\ 1 \leq i \leq n}} \frac{(1 - \gamma_i / \delta_i)}{(1 - q)} \prod_{\substack{i \in S \\ j \in S^c}} \frac{(1 - q)}{(1 - \delta_j / \delta_i)} \\ &\quad \times {}_\mu[\tilde{G}]_{m-1}^{(n)}(\gamma; \delta_i q^{\chi(i \in S)}; z), \end{aligned} \quad (2.5b)$$

where $\chi(i \in S)$ is 1 if $i \in S$, and 0 otherwise.

Proof. Applying (1.20), (1.2), and (1.5a) we obtain

$$\begin{aligned}
 {}_{\mu}[G]_m^{(n)}(\gamma; \delta; z) &= \frac{[l + \mu m]!}{[l - 1 + \mu(m - 1)]!} \\
 &\times \sum_{\substack{\|S\| = \mu + 1 \\ S \subseteq I_n}} \{ {}_0D_{l + \mu m}^{(n)}([A]; p) \}^{-1} \\
 &\times {}_0D_{l - 1 + \mu(m - 1)}^{(n)}([A] - [A(S)]; p) \\
 &\times \prod_{k \in S^c} z_k \prod_{\substack{i < j \\ i \in S, j \in S^c}} \frac{(1 - q)}{(1 - \gamma_i/q\gamma_j)} \prod_{\substack{i < j \\ i \in S^c, j \in S}} \frac{(1 - q)}{((\gamma_j/q\gamma_i) - 1)} \\
 &\times {}_{\mu}[G]_{m-1}^{(n)}(\gamma_i q^{-\chi(i \in S)}; \delta; z). \tag{2.6a}
 \end{aligned}$$

Applying formula (1.26) we obtain

$$\begin{aligned}
 {}_{\mu}[G]_m^{(n)}(\gamma; \delta; z) &= \sum_{\substack{\|S\| = \mu + 1 \\ S \subseteq I_n}} \prod_{\substack{j \in S \\ 1 \leq i < j}} \left(\frac{\gamma_j}{\delta_i q} \right) \prod_{i \in S} \frac{(1 - \gamma_i/\delta_i)}{(1 - q)} \\
 &\times \prod_{\substack{i \in S \\ i < j \leq n}} \frac{(1 - \gamma_i/\delta_j)}{(1 - q)} \prod_{\substack{j \in S \\ 1 \leq i < j}} \frac{(1 - \delta_i/\gamma_j)}{(1 - q)} \\
 &\times \prod_{\substack{i < j \\ i \in S, j \in S^c}} \frac{(1 - \gamma_i/q\gamma_j)}{(1 - \gamma_i/\gamma_j)} \prod_{\substack{i < j \\ i \in S^c, j \in S}} \frac{(1 - q\gamma_i/\gamma_j)}{(1 - \gamma_i/\gamma_j)} \\
 &\times \prod_{k \in S^c} z_k \prod_{\substack{i < j \\ i \in S, j \in S^c}} \frac{(1 - q)}{(1 - \gamma_i/q\gamma_j)} \prod_{\substack{i < j \\ i \in S^c, j \in S}} \frac{(1 - q)}{((\gamma_j/q\gamma_i) - 1)} \\
 &\times {}_{\mu}[G]_{m-1}^{(n)}(\gamma_i q^{-\chi(i \in S)}; \delta; z). \tag{2.6b}
 \end{aligned}$$

After some cancellation and easy manipulation one obtains (2.5a). Similarly the proof of equation (2.5b) follows from (1.21), (1.2), (1.5b), and (1.26) after an easy but lengthy computation. Q.E.D.

COROLLARY 2.7. *There are unique rational functions $R(\gamma; \delta; z)$ and $\tilde{R}(\gamma; \delta; z)$ in the variables $(\gamma_1, \dots, \gamma_n; \delta_1, \dots, \delta_n; z_1, \dots, z_n)$ which are equal to ${}_{\mu}[G]_m^{(n)}(\gamma; \delta; z)$ and ${}_{\mu}[\tilde{G}]_m^{(n)}(\gamma; \delta; z)$, respectively, for all $(\gamma; \delta; z)$ satisfying the conditions in Definition 2.1. We shall denote $R(\gamma; \delta; z)$ and $\tilde{R}(\gamma; \delta; z)$ by ${}_{\mu}[G]_m^{(n)}(\gamma; \delta; z)$ and ${}_{\mu}[\tilde{G}]_m^{(n)}(\gamma; \delta; z)$, respectively.*

Proof. Identities (2.5a) and (2.5b) are independent of the integer l in the constraint $\sum_{i=1}^n A_i = l + \mu m$. Thus by iterating (2.5a) and (2.5b) we obtain rational expressions for ${}_{\mu}[G]_m^{(n)}(\gamma; \delta; z)$ and ${}_{\mu}[\tilde{G}]_m^{(n)}(\gamma; \delta; z)$ in the

variables $(\gamma; \delta; z)$ which are valid whenever $[\Delta] \in W^n$, $\lambda \in \Lambda_n$, $z \in \mathbb{C}^n$, and $p_i > p_j + \Delta_j + 1$ for $1 \leq i < j \leq n$. One then observes that two rational functions in $(\gamma; \delta; z)$ must be equal if they agree for these values of $(\gamma; \delta; z)$.
Q.E.D.

As an immediate consequence of Corollary 2.7 and identities (2.5a–b) we have

COROLLARY 2.8. *Let $\sigma, \tau \in S_n$ be arbitrary permutations of the set $\{1, \dots, n\}$. Define $\gamma^\sigma = (\gamma_{\sigma(1)}, \dots, \gamma_{\sigma(n)})$ and $\delta^\tau = (\delta_{\tau(1)}, \dots, \delta_{\tau(n)})$ and similarly z^σ and z^τ . Then*

$${}_\mu[G]_m^{(n)}(\gamma^\sigma; \delta^\tau; z^\sigma) = {}_\mu[G]_m^{(n)}(\gamma; \delta; z) \quad (2.9a)$$

and

$${}_\mu[\tilde{G}]_m^{(n)}(\gamma^\sigma; \delta^\tau; z^\tau) = {}_\mu[\tilde{G}]_m^{(n)}(\gamma; \delta; z), \quad (2.9b)$$

where both sides of (2.9a–b) are viewed as rational functions in the variables $(\gamma; \delta; z)$. Moreover, if $\gamma^* = (\delta_1^{-1}, \dots, \delta_n^{-1})$ and $\delta^* = (\gamma_1^{-1}, \dots, \gamma_n^{-1})$, then

$${}_\mu[G]_m^{(n)}(\gamma^*; \delta^*; z) = {}_\mu[\tilde{G}]_m^{(n)}(\gamma; \delta; z), \quad (2.9c)$$

$${}_\mu[\tilde{G}]_m^{(n)}(\gamma^*; \delta^*; z) = {}_\mu[G]_m^{(n)}(\gamma, \delta, z), \quad (2.9d)$$

where both sides of (2.9c), (2.9d) are viewed as rational functions of $(\gamma; \delta; z)$.

Corollary 2.8, Definition 2.1, and Lemma 1.13 imply

PROPOSITION 2.10. *Let γ^* and δ^* be defined as in Corollary 2.8 and let $0 \leq \mu \leq n-1$ and $z \in \mathbb{C}^n$. Then the operators $B_{\mu; z}$ and B_0 commute in general position (see Definition 1.11) if and only if*

$${}_\mu[G]_1^{(n)}(\gamma^*; \delta^*; z) = {}_\mu[G]_1^{(n)}(\gamma; \delta; z) \quad (2.11)$$

as rational functions in the variables $(\gamma_1, \dots, \gamma_n; \delta_1, \dots, \delta_n)$. If (2.11) holds then, for all $m \geq 0$, we have

$${}_\mu[G]_m^{(n)}(\gamma^*; \delta^*; z) = {}_\mu[G]_m^{(n)}(\gamma; \delta; z) \quad (2.12)$$

as rational functions in the variables $(\gamma; \delta)$.

DEFINITION 2.13. When identity (2.12) holds we say the function ${}_\mu[G]_m^{(n)}(\gamma; \delta; z)$ has *transposition symmetry*.

By Proposition 2.10, to show that the ${}_\mu[G]_m^{(n)}(\gamma; \delta; z)$ functions have transposition symmetry for fixed μ , $0 \leq \mu \leq n-1$, and fixed $z \in \mathbb{C}^n$ and all

$m \geq 0$, it suffices to show that the function ${}_{\mu}[G]_1^{(n)}(\gamma; \delta; z)$ has transposition symmetry. We shall compute the functions ${}_{\mu}[G]_1^{(n)}(\gamma; \delta; z)$ in the case $z = (t, \dots, t)$ for any $t \in \mathbb{C}$ and show that in this case ${}_{\mu}[G]_1^{(n)}(\gamma; \delta; z)$ has transposition symmetry. We shall then show that when $z \neq (t, \dots, t)$ for some $t \in \mathbb{C}$ then ${}_{\mu}[G]_1^{(n)}(\gamma; \delta; z)$ does not have transposition symmetry (except in trivial cases when ${}_{\mu}[G]_1^{(n)}(\gamma; \delta; z) \equiv 0$).

PROPOSITION 2.14. *The function ${}_{\mu}[G]_1^{(n)}(\gamma; \delta; z)$ has transposition symmetry for fixed $z \in \mathbb{C}^n$ if and only if $z = (t, \dots, t)$ for some $t \in \mathbb{C}$ or $z = (z_1, \dots, z_n)$ with at least $\mu + 2$ of the $z_i = 0$ for $1 \leq i \leq n$.*

Proof. From Proposition 2.4 we have

$$\begin{aligned} {}_{\mu}[G]_1^{(n)}(\gamma; \delta; z) &= (-1)^{\binom{\mu+1}{2}} q^{-\binom{\mu+1}{2}} (1-q)^{-(\mu+1)^2} \\ &\quad \times \sum_{\substack{\|S\| = \mu+1 \\ S \subseteq I_n}} \prod_{k \in S^c} z_k \prod_{\substack{i \in S \\ 1 \leq i \leq n}} (1 - \gamma_i / \delta_i) \prod_{\substack{i \in S \\ j \in S^c}} (1 - \gamma_i / \gamma_j)^{-1} \end{aligned} \quad (2.15a)$$

$$\begin{aligned} &= (-1)^{\binom{\mu+1}{2}} q^{-\binom{\mu+1}{2}} (1-q)^{-(\mu+1)^2} \\ &\quad \times \sum_{\substack{\|S\| = \mu+1 \\ S \subseteq I_n}} \prod_{k \in S^c} z_k \prod_{i \in S} \gamma_i^{\mu+1} \prod_{\substack{i \in S \\ 1 \leq i \leq n}} (\gamma_i^{-1} - \delta_i^{-1}) \prod_{\substack{i \in S \\ j \in S^c}} (\gamma_i^{-1} - \gamma_j^{-1})^{-1}. \end{aligned} \quad (2.15b)$$

Replacing γ_i^{-1} by γ_i and δ_i^{-1} by δ_i , for $1 \leq i \leq n$, and multiplying by $q^{\binom{\mu+1}{2}} (1-q)^{(\mu+1)^2} \prod_{i=1}^n \gamma_i^{\mu+1}$ we obtain

$$\begin{aligned} U(\gamma; \delta; z) &\equiv q^{\binom{\mu+1}{2}} (1-q)^{(\mu+1)^2} \times \prod_{i=1}^n \gamma_i^{\mu+1} {}_{\mu}[G]_1^{(n)}(\gamma_j^{-1}, \dots, \delta_j^{-1}; z) \\ &= (-1)^{\binom{\mu+1}{2}} \sum_{\substack{\|S\| = \mu+1 \\ S \subseteq I_n}} \prod_{k \in S^c} z_k \gamma_k^{\mu+1} \prod_{\substack{i \in S \\ 1 \leq i \leq n}} (\gamma_i - \delta_i) \\ &\quad \times \prod_{\substack{i \in S \\ j \in S^c}} (\gamma_i - \gamma_j)^{-1}. \end{aligned} \quad (2.16)$$

Setting $\gamma^* = (\delta_1^{-1}, \dots, \delta_n^{-1})$ and $\delta^* = (\gamma_1^{-1}, \dots, \gamma_n^{-1})$ as above, it follows from (2.16) that transposition symmetry for ${}_{\mu}[G]_1^{(n)}(\gamma; \delta; z)$ is equivalent to the identity

$$U(\gamma; \delta; z) = \prod_{i=1}^n (\gamma_i \delta_i)^{\mu+1} U(\gamma^*; \delta^*; z) \quad (2.17)$$

as rational functions in the variables $(\gamma; \delta)$.

Before continuing with the proof of (2.17) we recall

DEFINITION 2.18. Let $(\gamma_1, \dots, \gamma_n) \in \mathbb{C}^n$ with $\gamma_i \neq \gamma_j$ for $1 \leq i < j \leq n$. Then the Vandermonde determinant is

$$V_n(\gamma) = V_n(\gamma_i) = \prod_{1 \leq i < j \leq n} (\gamma_i - \gamma_j). \quad (2.19)$$

If $\lambda = (\lambda_1, \dots, \lambda_n)$ is a partition, the Schur function of type λ is

$$S_\lambda(\gamma) = S_\lambda(\gamma_i) = \frac{1}{V_n(\gamma)} \sum_{w \in S_n} \varepsilon(w) \prod_{i=1}^n \gamma_{w(i)}^{\lambda_i + n - i}, \quad (2.20)$$

where \mathcal{S}_n is the symmetric group on I_n and $\varepsilon(w)$ is the sign of the permutation w .

To prove identity (2.17) we shall use a modified form of Theorem 1.20 of [12]. An easy consequence of the proof of Theorem 1.20 is the following.

LEMMA 2.21. For each $S \subseteq I_n$, $\|S\| = \mu + 1$, let $R_S(\gamma_i; \delta_i; z_i)$ be a rational function in the variables γ_i , δ_i and z_i for $1 \leq i \leq n$. For $w \in \mathcal{S}_n$, let $wS \subseteq I_n$ be the image of the set S under w . We further assume that

$$R_{wS}(\gamma_i; \delta_i; z_i) = R_S(\gamma_{w(i)}; \delta_{w(i)}; z_{w(i)}), \quad (2.22)$$

i.e., replacing γ_i , δ_i , z_i by $\gamma_{w(i)}$, $\delta_{w(i)}$, $z_{w(i)}$, respectively. We then have

$$\begin{aligned} & \sum_{\substack{S \subseteq I_n \\ \|S\| = \mu + 1}} (-1)^{\mu + 1 + \Sigma(S)} \prod_{\substack{1 \leq i < j \leq n \\ i \in S, j \in S^c}} (\gamma_i - \gamma_j)^{-1} \prod_{\substack{1 \leq i < j \leq n \\ i \in S^c, j \in S}} (\gamma_i - \gamma_j)^{-1} \\ & \times \prod_{\substack{i=1 \\ i \in S}}^n \prod_{\substack{l=1 \\ l \in S}}^n (\gamma_i - \delta_l) R_S(\gamma_i; \delta_l; z_i) \\ & = \frac{(-1)^{\binom{\mu+1}{2}}}{V_n(\gamma)} \sum_{\substack{\lambda = (\lambda_1, \dots, \lambda_{\mu+1}) \\ \lambda_i \leq n}} (-1)^{|\lambda|} S_{\lambda'}(\delta) \\ & \times \left\{ \sum_{w \in \mathcal{S}_n} \varepsilon(w) \prod_{i=1}^{\mu+1} \gamma_{w(i)}^{n + \mu + 1 - i - \lambda_{\mu+2-i}} \prod_{i=\mu+2}^n \gamma_{w(i)}^{n-i} \right. \\ & \left. \times R_{I_{\mu+1}}(\gamma_{w(i)}; \delta_{w(i)}; z_{w(i)}) \right\}, \quad (2.23) \end{aligned}$$

where $\Sigma(S) = \sum_{i \in S} i$, $\varepsilon(w)$ is the sign of the permutation w , $I_{\mu+1} = \{1, \dots, \mu+1\} \subseteq I_n$, λ is a partition and λ' is the conjugate partition of λ . That is, $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_{(\lambda_1)})$, with $\lambda'_i = \|\{j \mid \lambda_j \geq i\}\|$. Also $|\lambda| = \sum_{i=1}^{\mu+1} \lambda_i$ is the weight of the partition λ .

Applying Lemma 2.21 with $R_S(\gamma_i, \delta_i, z_i) = \prod_{k \in S^c} z_k \gamma_k^{(\mu+1)}$ we obtain

COROLLARY 2.24. *With notation as in (2.16) and (2.23) we have*

$$U(\gamma; \delta; z) = \frac{(-1)^{\binom{\mu+1}{2}}}{V_n(\gamma)} \sum_{\substack{\lambda = (\lambda_1, \dots, \lambda_{\mu+1}) \\ \lambda_1 \leq n}} (-1)^{|\lambda|} S_{\lambda}(\delta) \\ \times \left\{ \sum_{w \in \mathcal{S}_n} \varepsilon(w) \prod_{i=1}^{\mu+1} \gamma_{w(i)}^{n+\mu+1-i-\lambda_{(\mu+2-i)}} \prod_{i=\mu+2}^n \gamma_{w(i)}^{n+\mu+1-i} z_{w(i)} \right\}. \quad (2.25)$$

Proof. By Proposition 2.21 it suffices to show for all $S \subseteq I_n$ with $\|S\| = \mu + 1$ that

$$(-1)^{\mu+1+\Sigma(S)} \prod_{\substack{1 \leq i < j \leq n \\ i \in S, j \in S^c}} (\gamma_i - \gamma_j)^{-1} \prod_{\substack{1 \leq i < j \leq n \\ i \in S^c, j \in S}} (\gamma_i - \gamma_j)^{-1} \\ = (-1)^{\binom{\mu+1}{2}} \prod_{\substack{i \in S \\ j \in S^c}} (\gamma_i - \gamma_j)^{-1}. \quad (2.26)$$

Let B be the set of ordered pairs given by

$$B = \{(i, j) \mid 1 \leq i < j \leq n, i \in S^c, j \in S\}. \quad (2.27)$$

For each $(i, j) \in B$ we make the substitution $(-1)(\gamma_j - \gamma_i)$ for $(\gamma_i - \gamma_j)$ on the left-hand side of (2.26). We are then reduced to checking the agreement of signs on the left- and right-hand sides of (2.26).

From Eq. (2.11) of [1] we have

$$\|B\| = -\binom{\mu+2}{2} + \Sigma(S). \quad (2.28)$$

Since

$$\binom{\mu+2}{2} - (\mu+1) = \binom{\mu+1}{2}, \quad (2.29)$$

then the signs on both sides of (2.26) agree and equation (2.25) follows.

Q.E.D.

Continuing with the proof of identity (2.17) and Proposition 2.14 we shall need the following

DEFINITION 2.30. Let $\tau = (\tau_1, \dots, \tau_{\mu+1})$ be a partition with $\tau_1 \leq \mu + 1$. Let $x = \|\{i \mid \tau_i \geq i\}\|$. Then define

$$\alpha(\tau) = (\tau_1 + n - (\mu + 1), \dots, \tau_x + n - (\mu + 1), \tau_{x+1}, \dots, \tau_{\mu+1}) \quad (2.31)$$

and

$$\begin{aligned} \beta(\tau) = & (\mu + 1 - \tau_{\mu+1}, \dots, \mu + 1 - \tau_{x+1}, \mu + 1 - x, \dots, \mu + 1 - x, \\ & \dots, \mu + 1 - x, \mu + 1 - \tau_x, \dots, \mu + 1 - \tau_1) \end{aligned} \quad (2.32)$$

where $\beta(\tau)$ has n parts.

LEMMA 2.33. If $z = (t, \dots, t) \in \mathbb{C}^n$ with $t \in \mathbb{C}$ and $\lambda = (\lambda_1, \dots, \lambda_{\mu+1})$ is a partition with $\lambda_1 \leq n$, then

$$\begin{aligned} & \frac{1}{V_n(\gamma)} \sum_{w \in \mathcal{S}_n} \varepsilon(w) \prod_{i=1}^{\mu+1} \gamma_{w(i)}^{n+\mu+1-i-\lambda_{(\mu+2-i)}} \prod_{i=\mu+2}^n \gamma_{w(i)}^{n+\mu+1-i-z_{w(i)}} \\ &= \begin{cases} (-1)^{x(n-(\mu+1))} t^{n-(\mu+1)} S_{\beta(\tau)}(\gamma) & \text{if } \lambda = \alpha(\tau) \text{ for some } \mu+1 \\ & \text{part partition } \tau \text{ with } \tau_1 \leq \mu+1; \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (2.34)$$

Proof. If $\lambda = \alpha(\tau)$ as above, then

$$\begin{aligned} & \frac{1}{V_n(\gamma)} \sum_{w \in \mathcal{S}_n} \varepsilon(w) \prod_{i=1}^{\mu+1} \gamma_{w(i)}^{n+\mu+1-i-\lambda_{(\mu+2-i)}} \prod_{i=\mu+2}^n \gamma_{w(i)}^{n+\mu+1-i-t} \\ &= \frac{(-1)^{x(n-(\mu+1))} t^{n-(\mu+1)}}{V_n(\gamma)} \sum_{w \in \mathcal{S}_n} \varepsilon(w) \prod_{i=1}^{\mu+1-x} \gamma_{w(i)}^{\mu+1-\lambda_{(\mu+2-i)}+n-i} \\ & \quad \times \prod_{i=\mu+2-x}^{n-x} \gamma_{w(i)}^{\mu+1-x+n-i} \prod_{i=n+1-x}^n \gamma_{w(i)}^{n-\lambda_{(n+1-i)}+n-i} \end{aligned} \quad (2.35a)$$

$$= \frac{(-1)^{x(n-(\mu+1))} t^{n-(\mu+1)}}{V_n(\gamma)} \sum_{\omega \in \mathcal{S}_n} \varepsilon(\omega) \prod_{i=1}^n \gamma_{\omega(i)}^{\beta(\tau)_i + n - i} \quad (2.35b)$$

$$= (-1)^{x(n-(\mu+1))} t^{n-(\mu+1)} S_{\beta(\tau)}(\gamma). \quad (2.35c)$$

The sign $(-1)^{x(n-(\mu+1))}$ on the right-hand side of (2.35a) is the sign of the permutation moving the positions $i = \mu - x$ through $i = \mu + 1$ to the new positions $i = n + 1 - x$ through $i = n$.

On the other hand if there exists no $\mu + 1$ part partition τ with $\tau_1 \leq \mu + 1$ such that $\lambda = \alpha(\tau)$, then there must exist an integer j with $1 \leq j \leq \mu + 1$ such that

$$j \leq \lambda_j < n + j - (\mu + 1). \quad (2.36)$$

Otherwise if $x = \|\{i | \lambda_i \geq n + i - (\mu + 1)\}\|$ then $\tau = (\lambda_1 + \mu + 1 - n, \dots, \lambda_x + \mu + 1 - n, \lambda_{x+1}, \dots, \lambda_{\mu+1})$ is a partition and $\lambda = \alpha(\tau)$.

Now fix a j with $1 \leq j \leq \mu + 1$ satisfying (2.36). In the product on the left hand side of (2.34) the exponent of $\gamma_{w(\mu+2-j)}$ is equal to the exponent of $\gamma_{w(\mu+2+\lambda_j-j)}$. Noting that the $z_{w(i)} = t$ can be factored out, then (2.34) is an alternating sum with two equal exponents and hence vanishes. Q.E.D.

COROLLARY 2.37. *If $z = (t, \dots, t) \in \mathbb{C}^n$ then*

$$U(\gamma; \delta; z) = (-1)^{\binom{\mu+1}{2}} t^{n-(\mu+1)} \sum_{\substack{\tau = (\tau_1, \dots, \tau_{\mu+1}) \\ \tau_1 \leq \mu+1}} (-1)^{x(n-(\mu+1))} S_{\alpha(\tau)}(\delta) S_{\beta(\tau)}(\gamma), \quad (2.38)$$

where the τ are partitions and $x = \|\{i | \tau_i \geq i\}\|$.

The following lemma together with Corollary 2.37 will imply identity (2.17) in the case $z = (t, \dots, t) \in \mathbb{C}^n$.

LEMMA 2.39. *Let $\tau = (\tau_1, \dots, \tau_{\mu+1})$ be a partition with $\tau_1 \leq \mu + 1$ and $\tau' = (\tau'_1, \dots, \tau'_{\mu+1})$ be the conjugate partition. With notation as in (2.17) we have*

$$\prod_{i=1}^n (\gamma_i \delta_i)^{\mu+1} S_{\alpha(\tau)}(\delta^*) S_{\beta(\tau)}(\gamma^*) = S_{\alpha(\tau')}(\delta) S_{\beta(\tau')}(\gamma). \quad (2.40)$$

Proof. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition with $\lambda_1 \leq \mu + 1$. We have

$$V_n(\gamma_i^{-1}) = \prod_{1 \leq i \leq j \leq n} (\gamma_i^{-1} - \gamma_j^{-1}) \quad (2.41a)$$

$$= (-1)^{\binom{n}{2}} \left(\prod_{i=1}^n \gamma_i^{1-n} \right) V_n(\gamma_i). \quad (2.41b)$$

So

$$S_\lambda(\gamma_i^{-1}) \prod_{i=1}^n \gamma_i^{\mu+1} = \frac{1}{V_n(\gamma_i^{-1})} \sum_{w \in \mathcal{S}_n} \varepsilon(w) \prod_{i=1}^n \gamma_{w(i)}^{\mu+1+i-n-\lambda_i} \quad (2.42a)$$

$$= \frac{(-1)^{\binom{n}{2}}}{V_n(\gamma)} \sum_{w \in \mathcal{S}_n} \varepsilon(w) \prod_{i=1}^n \gamma_{w(i)}^{\mu+i-\lambda_i} \quad (2.42b)$$

$$= \frac{(-1)^{\binom{n}{2}} (-1)^{\lfloor \frac{n}{2} \rfloor}}{V_n(\gamma)} \sum_{w \in \mathcal{S}_n} \varepsilon(w) \prod_{i=1}^n \gamma_{w(i)}^{\mu+1-\lambda_{(n+1-i)}+n-i}, \quad (2.42c)$$

where from (2.42b) to (2.42c) we interchange i with $n - i + 1$ for $1 \leq i \leq n$ and $[n/2]$ is the greatest integer less than or equal to $n/2$

$$= S_{(\mu+1-\lambda_n, \dots, \mu+1-\lambda_1)}(\gamma) \quad (2.42d)$$

since $\binom{n}{2} + [n/2]$ is always even.

Let $\tau = (\tau_1, \dots, \tau_{\mu+1})$ be a partition with $\tau_1 \leq \mu + 1$. If

$$\alpha(\tau) = (\tau_1 + n - (\mu + 1), \dots, \tau_x + n - (\mu + 1), \tau_{x+1}, \dots, \tau_{\mu+1}) \quad (2.43)$$

as in Definition 2.30, then one checks that

$$\alpha(\tau)' = (\tau'_1, \tau'_2, \dots, \tau'_x, x, \dots, x, \tau'_{x+1}, \dots, \tau'_{\mu+1}), \quad (2.44)$$

where $\alpha(\tau)'$ has n -parts.

We observe that for any given i , $1 \leq i \leq \mu + 1$, we have $\tau_i \geq i$ if and only if $\tau'_i \geq i$. Hence $x = \|\{i | \tau_i \geq i\}\| = \|\{i | \tau'_i \geq i\}\|$. So

$$\begin{aligned} & (\mu + 1 - \alpha(\tau)'_n, \mu + 1 - \alpha(\tau)'_{n-1}, \dots, \mu + 1 - \alpha(\tau)'_1) \\ &= (\mu + 1 - \tau'_{\mu+1}, \dots, \mu + 1 - \tau'_{x+1}, \mu + 1 - x, \dots, \mu + 1 - x, \\ & \quad \mu + 1 - \tau'_x, \dots, \mu + 1 - \tau'_1) \end{aligned} \quad (2.45a)$$

$$= \beta(\tau'). \quad (2.45b)$$

Also

$$\begin{aligned} & (\mu + 1 - \beta(\tau)_n, \mu + 1 - \beta(\tau)_{n-1}, \dots, \mu + 1 - \beta(\tau)_1) \\ &= (\tau_1, \dots, \tau_x, x, \dots, x, \tau_{x+1}, \dots, \tau_{\mu+1}) \end{aligned} \quad (2.46a)$$

$$= \alpha(\tau)'. \quad (2.46b)$$

Identity (2.40) follows directly from equations (2.42d), (2.45b), and (2.46b). Q.E.D.

We now complete the proof of identity (2.17) in the case $z = (t, \dots, t) \in \mathbb{C}^n$. From Corollary 2.37 and Lemma 2.39 we have

$$\begin{aligned} & \prod_{i=1}^n (\gamma_i \delta_i)^{\mu+1} U(\gamma^*, \delta^*; z) \\ &= (-1)^{\binom{\mu+1}{2}} t^{n-(\mu+1)} \sum_{\substack{\tau = (\tau_1, \dots, \tau_{\mu+1}) \\ \tau_1 \leq \mu+1}} (-1)^{x(n-(\mu+1))} S_{\alpha(\tau)'}(\delta) S_{\beta(\tau)}(\gamma). \end{aligned} \quad (2.47a)$$

By interchanging τ' with τ and using $x = \|\{i \mid \tau_i \geq i\}\| = \|\{i \mid \tau'_i \geq i\}\|$, (2.47a) becomes

$$(-1)^{\binom{\mu+1}{2}} t^{n-(\mu+1)} \sum_{\substack{\tau = (\tau_1, \dots, \tau_{\mu+1}) \\ \tau_i \leq \mu+1}} (-1)^{x(n-(\mu+1))} S_{\alpha(\tau)}(\delta) S_{\beta(\tau)}(\gamma). \quad (2.47b)$$

With another application of Corollary 2.37 we obtain

$$U(\gamma; \delta; z). \quad (2.47c)$$

Q.E.D.

We will now complete the proof of Proposition 2.14. We suppose that for some fixed $z \in \mathbb{C}^n$ that ${}_{\mu}[G]_1^{(n)}(\gamma; \delta; z)$ has transposition symmetry. This is equivalent to the function $U(\gamma; \delta; z)$ having symmetry (2.17). From equation (2.25) it follows that $U(\gamma; \delta; z)$ is a symmetric function in the variables $(\delta_1, \dots, \delta_n) \in \mathbb{C}^n$ independently of the values of $(\gamma_1, \dots, \gamma_n) \in \mathbb{C}^n$. By transposition symmetry it follows that $U(\gamma; \delta; z)$ is symmetric in the variables $(\gamma_1, \dots, \gamma_n)$ independently of the values of $(\delta_1, \dots, \delta_n)$. Since the Schur functions $S_{\lambda}(\delta)$ in Eq. (2.25) are linearly independent over \mathbb{C} for distinct partitions $\lambda = (\lambda_1, \dots, \lambda_{\mu+1})$ with $\lambda_1 \leq n$ (see [14]), then for every partition with $\lambda_1 \leq n$ the expression

$$\frac{1}{V_n(\gamma)} \sum_{w \in \mathcal{S}_n} \varepsilon(w) \prod_{i=1}^{\mu+1} \gamma_{w(i)}^{n+\mu+1-i-\lambda_{(\mu+2-i)}} \prod_{i=\mu+2}^n \gamma_{w(i)}^{n+\mu+1-i} z_{w(i)} \quad (2.48)$$

is symmetric in the variables $(\gamma_1, \dots, \gamma_n) \in \mathbb{C}^n$. Alternatively, the expression

$$\sum_{w \in \mathcal{S}_n} \varepsilon(w) \prod_{i=1}^{\mu+1} \gamma_{w(i)}^{n+\mu+1-i-\lambda_{(\mu+2-i)}} \prod_{i=\mu+2}^n \gamma_{w(i)}^{n+\mu+1-i} z_{w(i)} \quad (2.49)$$

is skew-symmetric in the variables $(\gamma_1, \dots, \gamma_n) \in \mathbb{C}^n$ for every partition $\lambda = (\lambda_1, \dots, \lambda_{\mu+1})$ with $\lambda_1 \leq n$.

If we consider the partition $\lambda = (0, \dots, 0)$ then the monomials

$$\left\{ \prod_{i=1}^n \gamma_{w(i)}^{n+\mu+1-i} \mid w \in \mathcal{S}_n \right\}$$

are linearly independent over \mathbb{C} . Since the expression (2.49) is skew-symmetric then the products $\prod_{i=\mu+2}^n z_{w(i)}$ are equal for all $w \in \mathcal{S}_n$. This implies that these products are identically zero or that $z_1 = z_2 = \dots = z_n = t \in \mathbb{C}$. This completes the proof of Proposition 2.14. Q.E.D.

Remark 2.50. Recalling the notation of Corollary 2.8, we say that the

functions ${}_{\mu}[G]_m^{(n)}(\gamma; \delta; z)$, $m \geq 1$, are bisymmetric in the variables $\gamma = (\gamma_1, \dots, \gamma_n)$ and $\delta = (\delta_1, \dots, \delta_n)$ if for all permutations $\sigma, \tau \in \mathcal{S}_n$,

$${}_{\mu}[G]_m^{(n)}(\gamma^{\sigma}; \delta^{\tau}; z) = {}_{\mu}[G]_m^{(n)}(\gamma; \delta; z) \quad (2.51)$$

as rational functions in the γ and δ variables. The proof of Proposition 2.14 shows that the functions ${}_{\mu}[G]_m^{(n)}(\gamma; \delta; z)$ are bisymmetric in the variables γ and δ if and only if $z = (t, \dots, t) \in \mathbb{C}^n$ or $z = (z_1, \dots, z_n)$ with at least $\mu + 2$ of the z_i equal to zero. Thus requiring $z = (t, \dots, t)$ is a very natural condition since both bisymmetry and transposition symmetry are desirable properties of the ${}_{\mu}[G]_m^{(n)}(\gamma; \delta; z)$ functions.

3. ${}_{\mu}[G]_m^{(n)}$ FUNCTIONS

In this section we generalize the ${}_{\mu}[G]_m^{(n)}(\gamma; \delta; z)$ functions by allowing the number of δ variables $(\delta_1, \dots, \delta_v)$ to be v . If $v = n$ then we obtain the original ${}_{\mu}[G]_m^{(n)}$ functions. We shall prove that these new ${}_{\mu}[G]_m^{(n)}(\gamma; \delta; z)$ functions satisfy a pair of difference equations when $z = (t, \dots, t) \in \mathbb{C}^n$ and give a generalization of transposition symmetry.

DEFINITION 3.1. Let m, n, v, μ be nonnegative integers satisfying $n - 1 \geq \mu \geq 0$. Let $\{\gamma_1, \dots, \gamma_n; \delta_1, \dots, \delta_v\}$ be distinct nonzero complex numbers and $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. Given that ${}_{\mu}[G]_0^{(n)}(\gamma; \delta; z) \equiv 1$, we uniquely determine ${}_{\mu}[G]_m^{(n)}(\gamma; \delta; z)$ by means of

$$\begin{aligned} {}_{\mu}[G]_m^{(n)}(\gamma; \delta; z) &= {}_{\mu}[G]_m^{(n)}(\gamma_1, \dots, \gamma_n; \delta_1, \dots, \delta_v; z_1, \dots, z_n) \\ &= (-1)^{\binom{\mu+1}{2}} q^{-\binom{\mu+1}{2}} \sum_{\substack{\|S\| = \mu+1 \\ S \subseteq I_n}} \prod_{k \in S^c} z_k \\ &\quad \times \prod_{\substack{i \in S \\ 1 \leq i \leq v}} \frac{(1 - \gamma_i / \delta_i)}{(1 - q)} \prod_{\substack{i \in S \\ j \in S^c}} \frac{(1 - q)}{(1 - \gamma_i / \gamma_j)} \\ &\quad \times {}_{\mu}[G]_{m-1}^{(n)}(\gamma_i q^{-\chi(i \in S)}, \delta; z). \end{aligned} \quad (3.2)$$

The following proposition clarifies the relationship between the various ${}_{\mu}[G]_m^{(n)}(\gamma; \delta; z)$ functions.

PROPOSITION 3.3. *With assumptions as in Definition 3.1 and $v > 0$, $1 \leq s \leq v$, $1 \leq t \leq n$, then*

$$\begin{aligned} {}_{v-1}{}_{\mu}[G]_m^{(n)}(\gamma; \delta_1, \dots, \delta_s, \dots, \delta_v; z) \\ = (1 - q)^{m(\mu+1)} \lim_{\delta_s \rightarrow \infty} {}_{\mu}[G]_m^{(n)}(\gamma; \delta_1, \dots, \delta_s, \dots, \delta_v; z) \end{aligned} \quad (3.4a)$$

and

$$\begin{aligned} & {}_{\mu-1}^v[G]_m^{(n-1)}(\gamma_1, \dots, \hat{\gamma}_t, \dots, \gamma_n; \delta; z_1, \dots, \hat{z}_t, \dots, z_n) \\ &= (1-q)^{m(v-n+\mu+1)}(-q)^{\mu\mu} \lim_{\gamma_t \rightarrow 0} {}_{\mu}^v[G]_m^{(n)}(\gamma_1, \dots, \gamma_t, \dots, \gamma_n; \delta; z_1, \dots, z_t, \dots, z_n) \end{aligned} \quad (3.4b)$$

and

$$\begin{aligned} & (z_t)^m \times {}_{\mu}^{v-1}[G]_m^{(n-1)}(\gamma_1, \dots, \hat{\gamma}_t, \dots, \gamma_n; \delta_1, \dots, \hat{\delta}_s, \dots, \delta_v; z_1, \dots, \hat{z}_t, \dots, z_n) \\ &= {}_{\mu}^v[G]_m^{(n)}(\gamma_1, \dots, 1, \dots, \gamma_n; \delta_1, \dots, 1, \dots, \delta_v; z_1, \dots, z_t, \dots, z_n), \end{aligned} \quad (3.4c)$$

where $\hat{\delta}_s$, $\hat{\gamma}_t$, and \hat{z}_t mean to omit the δ_s , γ_t and z_t variables respectively and on the right-hand side of (3.4c) we set $\gamma_t = \delta_s = 1$.

Proof. By induction. Equations (3.4a)–(3.4c) are certainly true for $m=0$. If (3.4a) is true for $m=h$ then it follows for $m=h+1$ by taking limits as $\delta_s \rightarrow \infty$ on both sides of Eq. (3.2).

Now assume that (3.4b) is true for $m=h$. Then

$$\begin{aligned} & (1-q)^{(h+1)(v-n+\mu+1)}(-q)^{(h+1)\mu} \lim_{\gamma_t \rightarrow 0} {}_{\mu}^v[G]_{h+1}^{(n)}(\gamma; \delta; z) \\ &= (1-q)^{(v-n+\mu+1)}(-q)^{\mu}(-q)^{-\binom{\mu+1}{2}} \\ & \times \sum_{\substack{\|S\|=\mu+1 \\ S \subseteq I_n}} \lim_{\gamma_t \rightarrow 0} \left\{ \prod_{k \in S^c} z_k \prod_{\substack{i \in S \\ 1 \leq i \leq v}} \frac{(1-\gamma_i/\delta_i)}{(1-q)} \prod_{\substack{i \in S \\ j \in S^c}} \frac{(1-q)}{(1-\gamma_i/\gamma_j)} \right\} \\ & \times (1-q)^{h(v-n+\mu+1)}(-q)^{h\mu} \lim_{\gamma_t \rightarrow 0} {}_{\mu}^v[G]_h^{(n)}(\gamma_i q^{-\chi(i \in S)}, \delta; z). \end{aligned} \quad (3.5)$$

For each $S \subseteq I_n$, $\|S\| = \mu + 1$, if $t \in S$ then

$$\begin{aligned} & \lim_{\gamma_t \rightarrow 0} \left\{ \prod_{k \in S^c} z_k \prod_{\substack{i \in S \\ 1 \leq i \leq v}} \frac{(1-\gamma_i/\delta_i)}{(1-q)} \prod_{\substack{i \in S \\ j \in S^c}} \frac{(1-q)}{(1-\gamma_i/\gamma_j)} \right\} \\ &= (1-q)^{n-v-(\mu+1)} \prod_{k \in S^c} z_k \prod_{\substack{i \in S - \{t\} \\ 1 \leq i \leq v}} \frac{(1-\gamma_i/\delta_i)}{(1-q)} \\ & \times \prod_{\substack{i \in S - \{t\} \\ j \in S^c}} \frac{(1-q)}{(1-\gamma_i/\gamma_j)}. \end{aligned} \quad (3.6a)$$

On the other hand if $t \notin S$, then

$$\lim_{\gamma_t \rightarrow 0} \left\{ \prod_{k \in S^c} z_k \prod_{\substack{i \in S \\ 1 \leq i \leq v}} \frac{(1-\gamma_i/\delta_i)}{(1-q)} \prod_{\substack{i \in S \\ j \in S^c}} \frac{(1-q)}{(1-\gamma_i/\gamma_j)} \right\} = 0. \quad (3.6b)$$

Substituting (3.6a) and (3.6b) into Eq. (3.5) and applying the induction hypothesis together with (3.2) proves (3.4b) in the case $m = h + 1$.

Now assume that (3.4c) is true for $m = h$. Setting $\delta_s = \gamma_t = 1$ we obtain

$$\prod_{k \in S^c} z_k \prod_{\substack{i \in S \\ 1 \leq i \leq v}} \frac{(1 - \gamma_i / \delta_i)}{(1 - q)} \prod_{\substack{i \in S \\ j \in S^c}} \frac{(1 - q)}{(1 - \gamma_i / \gamma_j)} \\ = \begin{cases} 0 & \text{if } t \in S; \\ z_t \prod_{k \in S^c - \{t\}} z_k \prod_{\substack{i \in S \\ 1 \leq i \leq v, i \neq s}} \frac{(1 - \gamma_i / \delta_i)}{(1 - q)} \prod_{\substack{i \in S \\ j \in S^c - \{t\}}} \frac{(1 - q)}{(1 - \gamma_i / \gamma_j)} & \text{if } t \notin S. \end{cases} \quad (3.7)$$

Applying the induction hypothesis together with (3.2) and (3.7) gives (3.4c) in the case $m = h + 1$. Q.E.D.

In the case $z = (t, \dots, t) \in \mathbb{C}^n$ we have transposition symmetry for the function ${}_{\mu}[G]_m^{(n)}(\gamma; \delta; z)$. This means that ${}_{\mu}[G]_m^{(n)}(\gamma; \delta; z)$ also satisfies the difference equation (2.5b). For $n - (\mu + 1) < v$ we have

PROPOSITION 3.8. *Let $z = (t, \dots, t) \in \mathbb{C}^n$, $n - (\mu + 1) < v$ and otherwise assumptions as in Definition 3.1. Then*

$$\begin{aligned} & {}_v^v[G]_m^{(n)}(\gamma_1, \dots, \gamma_n; \delta_1, \dots, \delta_v; z) \\ &= (-1)^{\binom{\mu+1}{2}} q^{-\binom{\mu+1}{2}} t^{n - (\mu+1)} \\ & \times \sum_{\substack{\|S\| = \mu+1+v-n \\ S \subseteq I_v}} \prod_{\substack{i \in S \\ 1 \leq i \leq n}} \frac{(1 - \gamma_i / \delta_i)}{(1 - q)} \prod_{\substack{i \in S \\ j \in S^c}} \frac{(1 - q)}{(1 - \delta_j / \delta_i)} \\ & \times {}_v^v[G]_{m-1}^{(n)}(\gamma; \delta, q^{X(i \in S)}; z). \end{aligned} \quad (3.9)$$

Proof. From Propositions 2.10 and 2.14 ${}_{\mu}[G]_m^{(n)}(\gamma; \delta_1, \dots, \delta_n; z)$ has transposition symmetry. From identity (2.9c) it follows that ${}_{\mu}[G]_m^{(n)}(\gamma; \delta_1, \dots, \delta_n; z)$ satisfies difference equation (2.5b). If $v = n$ then equation (3.9) is identical to (2.5b). The proof of (3.9) will be divided into two cases, depending on whether $v \leq n$ or $v \geq n$.

Case I ($v \leq n$). Fixing n , assume that Proposition 3.8 is true for $v = h \leq n$. We shall demonstrate it for $v = h + 1 \leq n$. By induction this will complete the proof of (3.9) when $v \leq n$.

Applying (3.4a) to the $v = h$ case of (3.9) we find

$$\begin{aligned} h^{-1} [G]_m^{(n)}(\gamma; \delta_1, \dots, \delta_{h-1}; z) &= (-1)^{\binom{\mu+1}{2}} q^{-\binom{\mu+1}{2}} t^{n-(\mu+1)} \\ &\times (1-q)^{m(\mu+1)} \lim_{\delta_h \rightarrow \infty} \left\{ \sum_{\substack{\|S\| = \mu+1+h-n \\ S \subseteq I_h}} \prod_{\substack{i \in S \\ 1 \leq l \leq n}} \frac{(1-\gamma_l/\delta_i)}{(1-q)} \right. \\ &\times \left. \prod_{\substack{i \in S \\ j \in S^c}} \frac{(1-q)}{(1-\delta_j/\delta_i)} \times {}^h[G]_{m-1}^{(n)}(\gamma; \delta_i q^{x(i \in S)}; z) \right\} \quad (3.10a) \\ &= (-1)^{\binom{\mu+1}{2}} q^{-\binom{\mu+1}{2}} t^{n-(\mu+1)} (1-q)^{\mu+1} \end{aligned}$$

$$\begin{aligned} &\times \sum_{\substack{\|S\| = \mu+1+h-n \\ S \subseteq I_h}} \lim_{\delta_h \rightarrow \infty} \left\{ \prod_{\substack{i \in S \\ 1 \leq l \leq n}} \frac{(1-\gamma_l/\delta_i)}{(1-q)} \prod_{\substack{i \in S \\ j \in S^c}} \frac{(1-q)}{(1-\delta_j/\delta_i)} \right\} \\ &\times (1-q)^{(m-1)(\mu+1)} \lim_{\delta_h \rightarrow \infty} {}^h[G]_{m-1}^{(n)}(\gamma; \delta_i q^{x(i \in S)}; z). \quad (3.10b) \end{aligned}$$

For each $S \subseteq I_h$, $\|S\| = \mu+1+h-n$, we have

$$\begin{aligned} &\lim_{\delta_h \rightarrow \infty} \left\{ \prod_{\substack{i \in S \\ 1 \leq l \leq n}} \frac{(1-\gamma_l/\delta_i)}{(1-q)} \prod_{\substack{i \in S \\ j \in S^c}} \frac{(1-q)}{(1-\delta_j/\delta_i)} \right\} \\ &= \begin{cases} 0 & \text{if } h \notin S; \\ (1-q)^{-(\mu+1)} \prod_{\substack{i \in S - \{h\} \\ 1 \leq l \leq n}} \frac{(1-\gamma_l/\delta_i)}{(1-q)} \prod_{\substack{i \in S - \{h\} \\ j \in S^c}} \frac{(1-q)}{(1-\delta_j/\delta_i)} & \text{if } h \in S. \end{cases} \quad (3.11) \end{aligned}$$

Applying (3.4a) and (3.11) to Eq. (3.10b) we obtain identity (3.9) in the case $v = h-1$.

Case II ($v \geq n$). Fixing v , assume that Proposition 3.8 is true for $n = h \leq v$. We shall demonstrate it for $n = h-1 \leq v$. By induction this will complete the proof of (3.9) when $v \geq n$.

Applying identity (3.4b) to the $n = h$ and $\mu+1$ case of (3.9), we find

$$\begin{aligned} &{}^v[G]_m^{(h-1)}(\gamma_1, \dots, \gamma_{h-1}; \delta_1, \dots, \delta_v; z_1, \dots, z_{h-1}) \\ &= (-q)^{-\binom{\mu+2}{2}} (-q)^{\mu+1} t^{h-(\mu+2)} (1-q)^{(v-h+\mu+2)} \\ &\times \sum_{\substack{\|S\| = \mu+2+v-h \\ S \subseteq I_v}} \lim_{\gamma_h \rightarrow 0} \left\{ \prod_{\substack{i \in S \\ 1 \leq l \leq h}} \frac{(1-\gamma_l/\delta_i)}{(1-q)} \prod_{\substack{i \in S \\ j \in S^c}} \frac{(1-q)}{(1-\delta_j/\delta_i)} \right\} \\ &\times (1-q)^{(m-1)(v-h+\mu+2)} (-q)^{(m-1)(\mu+1)} \\ &\times \lim_{\gamma_h \rightarrow 0} {}^v[G]_{m-1}^{(h)}(\gamma; \delta_i q^{x(i \in S)}; z). \quad (3.12) \end{aligned}$$

Note that $\binom{\mu+2}{2} - (\mu+1) = \binom{\mu+1}{2}$ and $h - (\mu+2) = h-1 - (\mu+1)$. Apply to Eq. (3.12) the identities (3.4b) and the following equation:

$$\begin{aligned} \lim_{\gamma_h \rightarrow 0} \left\{ \prod_{\substack{i \in S \\ 1 \leq l \leq h}} \frac{(1 - \gamma_l / \delta_i)}{(1 - q)} \prod_{\substack{i \in S \\ j \in S^c}} \frac{(1 - q)}{(1 - \delta_j / \delta_i)} \right\} \\ = (1 - q)^{h - v - (\mu+2)} \left\{ \prod_{\substack{i \in S \\ 1 \leq l \leq h-1}} \frac{(1 - \gamma_l / \delta_i)}{(1 - q)} \prod_{\substack{i \in S \\ j \in S^c}} \frac{(1 - q)}{(1 - \delta_j / \delta_i)} \right\}, \end{aligned} \quad (3.13)$$

where $S \subseteq I_v$ and $\|S\| = \mu + 2 + v - h$. We obtain identity (3.9) in the case $n = h - 1$. Q.E.D.

Remark 3.14. As a consequence of Proposition 3.3 or Definition 3.1 there is an obvious generalization of the symmetry (2.9a) to the ${}^\nu_\mu[G]_m^{(n)}(\gamma; \delta; z)$ functions. When $z = (t, \dots, t) \in \mathbb{C}^n$ then the fact that the ${}^\nu_\mu[G]_m^{(n)}(\gamma; \delta; z)$ functions satisfy two distinct q -difference equations, (3.2) and (3.9), is equivalent to a generalization (Proposition 3.15) of the transposition symmetry of Proposition 2.14.

PROPOSITION 3.15. *Let $z = (t, \dots, t) \in \mathbb{C}^n$, $z' = (t, \dots, t) \in \mathbb{C}^v$, $n - (\mu + 1) < v$ and otherwise assumptions as in Definition 3.1. Then*

$$\begin{aligned} {}^\nu_\mu[G]_m^{(n)}(\gamma_1, \dots, \gamma_n; \delta_1, \dots, \delta_v; z) \\ = (-q)^{[(\mu+1+\frac{1}{2}v-n) - (\frac{\mu+1}{2})]m} {}_{\mu+v-n}^n[G]_m^{(v)}(\delta_1^{-1}, \dots, \delta_v^{-1}; \gamma_1^{-1}, \dots, \gamma_n^{-1}; z') \end{aligned} \quad (3.16)$$

as rational functions in the variables $\gamma_1, \dots, \gamma_n; \delta_1, \dots, \delta_v$.

Proof. By induction on m . Equation (3.16) is certainly true when $m = 0$. We shall assume that it is true for $m = k \geq 0$ and prove it for $m = k + 1$.

From Proposition 3.8 we have

$$\begin{aligned} (-q)^{[(\mu+1+\frac{1}{2}v-n) - (\frac{\mu+1}{2})](k+1)} {}_{\mu+v-n}^n[G]_{k+1}^{(v)}(\delta_i^{-1}, \gamma_i^{-1}, ; z') \\ = (-q)^{-(\frac{\mu+1}{2})} t^{v - (\mu+1+v-n)} \sum_{\substack{\|S\| = \mu+1 \\ S \subseteq I_n}} \prod_{\substack{i \in S \\ 1 \leq l \leq v}} \frac{(1 - \gamma_l / \delta_l)}{(1 - q)} \\ \times \prod_{\substack{i \in S \\ j \in S^c}} \frac{(1 - q)}{(1 - \gamma_i / \gamma_j)} \times (-q)^{[(\mu+1+\frac{1}{2}v-n) - (\frac{\mu+1}{2})]k} \\ \times {}_{\mu+v-n}^n[G]_k^{(v)}(\delta_i^{-1}, (\gamma_i q^{-\chi(i \in S)})^{-1}, ; z'). \end{aligned} \quad (3.17)$$

Under the induction hypothesis the right-hand side of (3.17) is identical to the q -difference equation (3.2) for ${}^\nu_\mu[G]^{(n)}_{k+1}(\gamma; \delta; z)$. This proves (3.16) in the case $m = k + 1$. Q.E.D.

In a manner similar to Corollary 2.24 one can compute the first iteration, ${}^\nu_\mu[G]^{(n)}_1(\gamma; \delta; z)$, of the q -difference equation (3.2). This computation is used to give a uniqueness result for the transposition symmetry of the ${}^\nu_\mu[G]^{(n)}_1(\gamma; \delta; z)$ functions similar to Proposition 2.14. When $z = (t, \dots, t) \in \mathbb{C}^n$ we will also find an expression for ${}^\nu_\mu[G]^{(n)}_1(\gamma; \delta; z)$ in terms of Schur functions in the variables γ and δ .

DEFINITION 3.18. Let $n - (\mu + 1) < \nu$ and $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. Define

$$\begin{aligned} {}^\nu U^{(n)}(\gamma; \delta; z) &\equiv q^{\binom{\mu+1}{2}} (1-q)^{(\mu+1)(\mu+1+\nu-n)} \\ &\quad \times \prod_{i=1}^n \gamma_i^{\mu+1+\nu-n} \times {}^\nu_\mu[G]^{(n)}_1(\gamma_j^{-1}, \delta_j^{-1}; z) \\ &= (-1)^{\binom{\mu+1}{2}} \sum_{\substack{\|S\| = \mu+1 \\ S \subseteq I_n}} \prod_{k \in S^c} z_k \gamma_k^{\mu+1+\nu-n} \prod_{\substack{i \in S \\ 1 \leq i \leq \nu}} (\gamma_i - \delta_i) \\ &\quad \times \prod_{\substack{i \in S \\ j \in S^c}} (\gamma_i - \gamma_j)^{-1}. \end{aligned} \quad (3.19)$$

The following proposition is a generalization of Corollary 2.24. The proof is entirely similar except that $\mu + 1 + \nu - n$ is substituted in place of $\mu + 1$ in the exponents of the γ variables in Eq. (2.25) and in the corresponding generalization of Lemma 2.21.

PROPOSITION 3.20. Let $n - (\mu + 1) < \nu$ and $z = (z_1, \dots, z_n) \in \mathbb{C}^n$. Then

$$\begin{aligned} {}^\nu U^{(n)}(\gamma; \delta; z) &= \frac{(-1)^{\binom{\mu+1}{2}}}{V_n(\gamma)} \sum_{\substack{\lambda = (\lambda_1, \dots, \lambda_{\mu+1}) \\ \lambda_1 \leq \nu}} (-1)^{|\lambda|} S_{\lambda}(\delta) \\ &\quad \times \left\{ \sum_{w \in \mathcal{S}_n} e(w) \prod_{i=1}^{\mu+1} \gamma_{w(i)}^{\nu+\mu+1-i-\lambda_{i(\mu+2-i)}} \prod_{i=\mu+2}^n \gamma_{w(i)}^{\nu+\mu+1-i} z_{w(i)} \right\}. \end{aligned} \quad (3.21)$$

As a corollary we prove a uniqueness condition for transposition symmetry (3.16).

COROLLARY 3.22. Fix $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and $z' = (z'_1, \dots, z'_\nu) \in \mathbb{C}^\nu$. Let $n - (\mu + 1) < \nu$ and otherwise assumptions as in Definition 3.1. Then

$$\begin{aligned} &{}^\nu_\mu[G]^{(n)}_1(\gamma_1, \dots, \gamma_n; \delta_1, \dots, \delta_\nu; z) \\ &\equiv (-q)^{[(\mu+1+\nu-n) - \binom{\mu+1}{2}]} {}^\nu_{\mu+\nu-n}[G]^{(\nu)}_1(\delta_1^{-1}, \dots, \delta_\nu^{-1}; \gamma_1^{-1}, \dots, \gamma_n^{-1}; z') \end{aligned} \quad (3.23)$$

as rational functions in the variables $\gamma_1, \dots, \gamma_n; \delta_1, \dots, \delta_v$ if and only if $z = (t, \dots, t) \in \mathbb{C}^n$ and $z' = (t', \dots, t') \in \mathbb{C}^v$, where $t^{n-(\mu+1)} = (t')^{n-(\mu+1)}$, or both sides of (3.23) are identically zero.

Proof. When $z = (t, \dots, t) \in \mathbb{C}^n$ and $z' = (t', \dots, t') \in \mathbb{C}^v$ then Proposition 3.15 shows that (3.23) is satisfied. Equation (3.21) shows that when $t^{n-(\mu+1)} = (t')^{n-(\mu+1)}$ then the right-hand sides of (3.23) are equal for $z = (t, \dots, t) \in \mathbb{C}^n$ and $z' = (t', \dots, t') \in \mathbb{C}^v$.

On the other hand if (3.23) is true then, just as in the proof of Proposition 2.14, it follows that ${}^\nu_\mu[G]_1^{(n)}(\gamma; \delta; z)$ and ${}_{\mu+v-n}^\eta[G]_1^{(v)}(\delta_1^{-1}, \dots, \delta_v^{-1}; \gamma_1^{-1}, \dots, \gamma_n^{-1}; z')$ are both bisymmetric in the variables γ and δ . As before this implies that unless both sides of (3.23) are identically zero then $z = (t, \dots, t) \in \mathbb{C}^n$ and $z' = (t', \dots, t') \in \mathbb{C}^v$. An application of Proposition 3.15 then shows that

$${}^\nu_\mu[G]_1^{(n)}(\gamma; \delta; t, \dots, t) \equiv {}^\nu_\mu[G]_1^{(n)}(\gamma; \delta; t', \dots, t') \quad (3.24)$$

as rational functions in the variables γ and δ . By considering the $\lambda = (0, \dots, 0)$ term in (3.21) one checks that this is possible only when the coefficients of the monomial $\prod_{i=1}^n \gamma_i^{v+\mu+1-i}$ are equal, i.e., $t^{n-(\mu+1)} = (t')^{n-(\mu+1)}$. Q.E.D.

We now compute ${}^\nu_\mu[G]_1^{(n)}(\gamma; \delta; t, \dots, t)$. The computation is almost identical to that of Corollary 2.37 except that $\mu + 1 + v - n$ is substituted in place of $\mu + 1$ in the exponents of the γ variables.

DEFINITION 3.25. Let $\tau = (\tau_1, \dots, \tau_{\mu+1})$ be a partition with $\tau_1 \leq \mu + 1 + v - n$. Let

$$x = \|\{i \mid \tau_i \geq i\}\|. \quad (3.26)$$

Then define

$$\alpha(\tau) = (\tau_1 + n - (\mu + 1), \dots, \tau_x + n - (\mu + 1), \tau_{x+1}, \dots, \tau_{\mu+1}) \quad (3.27a)$$

and

$$\begin{aligned} \beta(\tau) = & (\mu + 1 + v - n - \tau_{\mu+1}, \dots, \mu + 1 + v - n - \tau_{x+1}, \mu + 1 + v - n - x, \\ & \dots, \mu + 1 + v - n - x, \mu + 1 + v - n - \tau_x, \dots, \mu + 1 + v - n - \tau_1) \end{aligned} \quad (3.27b)$$

where $\beta(\tau)$ has n parts.

PROPOSITION 3.28. *Let $n - (\mu + 1) < v$ and notation as in Definition 3.25. If $z = (t, \dots, t) \in \mathbb{C}^n$ then*

$$\begin{aligned} & {}^v[G]_1^{(n)}(\gamma; \delta; z) \\ &= (-q)^{-\binom{\mu+1}{2}} t^{n-(\mu+1)} (1-q)^{-(\mu+1)(\mu+1+v-n)} \prod_{i=1}^n \gamma_i^{\mu+1+v-n} \\ & \times \sum_{\substack{\tau = (\tau_1, \dots, \tau_{\mu+1}) \\ \tau_1 \leq \mu+1+v-n}} (-1)^{x(n-(\mu+1))} S_{\alpha(\tau)}(\delta_j^{-1},) S_{\beta(\tau)}(\gamma_j^{-1},) \quad (3.29) \end{aligned}$$

where the τ are partitions.

Proof. If one follows the proof of (2.38) starting from equation (2.25) then, with simple modifications, this gives a computation of ${}^vU^{(n)}(\gamma; \delta; z)$ starting from Eq. (3.21). The major difference in the proof comes from replacing the $\mu + 1$ terms appearing in the exponents of the γ variables by $\mu + 1 + v - n$. The final step is to apply (3.19) to compute ${}^v_\mu[G]_1^{(n)}(\gamma; \delta; z)$ from ${}^vU^{(n)}(\gamma; \delta; z)$. Q.E.D.

Remark 3.30. Set $z = (1, \dots, 1) \in \mathbb{C}^n$ and $\gamma_i = q^{\gamma_i}$, $\delta_j = q^{\delta_j}$ for $1 \leq i \leq n$ and $1 \leq j \leq v$. Then

$$\lim_{q \rightarrow 1} {}^v_\mu[G]_m^{(n)}(q^{\gamma_i};, q^{\delta_j}; z) = {}^v_\mu G_m^{(n)}(\gamma_i; , \delta_j;), \quad (3.31)$$

the ordinary G -function defined in Definition 1.17 of [12].

Comparing the expression (3.29) for ${}^v_\mu[G]_1^{(n)}(\gamma; \delta; z)$ with the expression (1.23) of [12] for ${}^v_\mu G_1^{(n)}(\gamma; \delta)$, they are both sums of products of a Schur function in the γ variables with a Schur function in the δ variables. The indices of summation vary over the same set of partitions. Despite this remarkable similarity, the limit of one term in (3.29) is not in general one term in (1.23) of [12]. In other words, the limit of the sum (3.29) is the expression (1.23) of [12] but not term by term.

Remark 3.32. We also remark that the ${}^v_\mu[G]_m^{(n)}(\gamma; \delta; z)$ here are equivalent but not identical to the ${}^v_\mu[G]_m^{(n)}(\gamma; \delta | (z))$ functions in [17]. One reason for this change is to facilitate the different types of computation in each paper.

4. THE $B_{\mu;z}$ OPERATORS AND RACAH-WIGNER ALGEBRA

The purpose of this section is to briefly motivate Definition 1.1 of the $B_{\mu;z}$ operators in terms of a “ q -analog” of the Racah-Wigner algebra of ten-

sor operators of the unitary groups. We give a simple definition of the “ q -analog” of the elementary reduced Wigner coefficients for the unitary group $U(n)$. This is done by replacing the fundamental concept of the measure of a Young tableau by a “ q -analog” of the measure. A more complete discussion of the construction and properties of the q -analog of the Racah–Wigner algebra is the subject of [11].

The definition of the $B_{\mu; z}$ operators given in formula (1.2) is simply a q -analog of the Biedenharn path sum formula. The ordinary path sum formula was stated and proved in the $n=3$ case, $z=(1, 1, 1)$, in [3]. For general n , $z=(1, \dots, 1) \in \mathbb{C}^n$, a statement of the ordinary path sum formula was made in [1]. Finally, the complete proof for general n , $z=(1, \dots, 1) \in \mathbb{C}^n$, was published in [9].

The ordinary path-sum formula has an elegant, though highly technical interpretation as an asymptotic limit of a reduced Wigner coefficient for the unitary group $U(n)$ (see [9]). This leads to the question: If there is a q -analog of the asymptotic limit, is there a q -analog of the reduced Wigner coefficient? The answer is given by

DEFINITION 4.1. (see Eq. (38) of [7]). Let $[m]_n = (m_{1n}, \dots, m_{nn})$ and $[m]_{n-1} = (m_{1n-1}, \dots, m_{n-1n-1})$ be partitions satisfying the betweenness conditions

$$m_{in} \leq m_{in-1} \leq m_{i+1n} \quad (4.2)$$

for $1 \leq i \leq n-1$. Let $p_{in} = m_{in} + n - i$ and $p_{jn-1} = m_{jn-1} + n - 1 - j$ for $1 \leq i \leq n$ and $1 \leq j \leq n-1$. Let k be an integer, $1 \leq k \leq n-1$, and $i_1, \dots, i_k; j_1, \dots, j_k$ be integers satisfying

$$1 \leq i_1 < i_2 < \dots < i_k \leq n \quad (4.3a)$$

and

$$1 \leq j_1 < j_2 < \dots < j_k \leq n-1. \quad (4.3b)$$

The weights $\Delta(i_1, i_2, \dots, i_k)$ (resp. $\Delta(j_1, \dots, j_k)$) is the n -tuple ($n-1$ tuple) with ones at the i_1, \dots, i_k (j_1, \dots, j_k) positions and zeros elsewhere. Let $0 < q$ and $q \neq 1$. With notation as in [7] we define the elementary reduced “ q -Wigner” coefficient by

$$\begin{aligned} & \left\langle \left(\begin{matrix} [m]_n + \Delta(i_1, i_2, \dots, i_k) \\ [m]_{n-1} + \Delta(j_1, j_2, \dots, j_k) \end{matrix} \right) \left| \begin{matrix} (i_1 i_2 \dots i_k) \\ [1_k 0_{n-k}] \\ (j_1, j_2, \dots, j_k) \end{matrix} \right| \left(\begin{matrix} [m]_n \\ [m]_{n-1} \end{matrix} \right) \right\rangle^2 \\ &= \left| \prod_{l=1}^k \left\{ \prod_{\substack{i=1 \\ i \neq (i_1 \dots i_k)}}^n \frac{(1 - q^{(p_{l,n-1} - p_{in} + 1)})}{(1 - q^{(p_{l,n} - p_{in})})} \prod_{\substack{j=1 \\ j \neq (j_1 \dots j_k)}}^{n-1} \frac{(1 - q^{(p_{l,n} - p_{j,n-1})})}{(1 - q^{(p_{l,n-1} - p_{j,n-1} + 1)})} \right\} \right|, \end{aligned} \quad (4.4)$$

where $[1_k 0_{n-k}]$ is the n -tuple with k ones followed by $n-k$ zeros. The sign of the square root is determined by a phase convention.

In [9] the elementary reduced Wigner coefficients are constructed via the product structure on the Weyl algebra of creation and annihilation operators. Conversely, once the elementary reduced Wigner coefficients are given then the multiplicative structure of the algebra of creation and annihilation operators is determined. Similarly, if the elementary reduced “ q -Wigner” coefficients are determined then it is possible to define a “ q -analog” of the algebra of creation and annihilation operators (see [11]).

In the ordinary case a combinatorial “pattern calculus” was developed by Biedenharn and Louck [7] to compute the elementary reduced Wigner coefficients. Underlying this pattern calculus is the notion of a “measure” of a Young tableau [10]. By redefining this “measure” one can obtain a theory which is structurally similar to the ordinary Racah–Wigner algebra, but whose building blocks are “ q -analogs” of the elementary reduced Wigner coefficients.

We will define below the measure \mathcal{M}_λ and its “ q -analog” $\mathcal{M}_\lambda(q)$ of a Young tableau of shape λ and weight λ , where $\lambda = (\lambda_1, \dots, \lambda_n)$ is a partition. (For definitions see [14].)

DEFINITION 4.5. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition and

$$N = |\lambda| = \sum_{i=1}^n \lambda_i. \quad (4.6)$$

We shall say an ordered pair of integers $x = (i, j)$ is a node of λ , denoted by $x \in \lambda$, if $1 \leq i \leq n$ and $1 \leq j \leq \lambda_i$. For each node $x \in \lambda$ define the hook length $h(x)$ by

$$h(x) = h(i, j) = \lambda_i + \lambda'_j - i - j + 1, \quad (4.7)$$

where $\lambda' = (\lambda'_1, \dots, \lambda'_{\lambda_1})$ is the conjugate partition to λ .

Let f^λ be the degree of the irreducible representation of the symmetric group S_N associated to λ (see Robinson [18]). Equivalently f^λ is the number of standard Young tableaux of shape λ . Then define the measure

$$\mathcal{M}_\lambda = \frac{N!}{f^\lambda} = \prod_{x \in \lambda} h(x) \quad (4.8a)$$

and the “ q -measure”, for $0 < q < 1$,

$$\mathcal{M}_\lambda(q) = \prod_{x \in \lambda} (1 - q^{h(x)}). \quad (4.8b)$$

Remark 4.9. The measure $\mathcal{M}_\lambda(q)$ is equal to $H_\lambda(q)$, the “hook polynomial” (see p. 28 of [14]). The hook polynomial occurs in some important combinatorial generating functions. If

$$n(\lambda) = \sum_{i=1}^n (i-1) \lambda_i \quad (4.10)$$

then the generating function for all column-strict plane partitions of shape λ is (Eq. (1), p. 49 of [14])

$$q^{|\lambda| + n(\lambda)} H_\lambda(q)^{-1}. \quad (4.11)$$

If

$$\phi_n(q) = (1-q)(1-q^2) \cdots (1-q^n), \quad (4.12)$$

then the generating function for the number of standard tableaux T of shape λ is (Eq. (3), p. 50 [14])

$$\sum_T q^{r(T)} = q^{n(\lambda)} \phi_n(q) H_\lambda(q)^{-1}, \quad (4.13)$$

where

$$r(T) = \|\{1 \leq k \leq N \mid k+1 \text{ lies in a lower row than } k\}\|. \quad (4.14)$$

There are a number of other interesting identities involving the hook polynomial on pp. 50–53 of [14].

We finally remark that $\mathcal{M}_\lambda(q) = H_\lambda(q)$ has a beautiful interpretation in terms of the degrees of the distinct irreducible representations of $GL_N(F_q)$, where q is a prime power, occurring as components of the representation of $GL_N(F_q)$ induced from the trivial character on the upper triangular subgroup. These distinct irreducible components are in one-to-one correspondence with the partitions λ with arbitrary number of parts such that $|\lambda| = N$ (see [18] or [14]). Their degrees are given by

$$d_\lambda = q^{n(\lambda')} \phi_N(q) H_\lambda(q)^{-1} \quad (4.15)$$

(see equation (6.7) of [14]).

REFERENCES

1. L. BIEDENHARN, W. HOLMAN III, AND S. MILNE, The invariant polynomials characterizing $U(n)$ tensor operators $\langle p, q, \dots, q, 0, \dots, 0 \rangle$ having maximal null space, *Advan. Appl. Math.* **1** (1980), 390–472.
2. L. C. BIEDENHARN, R. A. GUSTAFSON, AND S. C. MILNE, An umbral calculus for polynomials characterizing $U(n)$ tensor operators, *Advan. in Math.* **51** (1984), 36–90.

3. L. C. BIEDENHARN, J. D. LOUCK, E. CHACON, AND M. CIFTAN, On the structure of the canonical tensor operators in the unitary groups. I. An extension of the pattern calculus rules and the canonical spitting in $U(3)$, *J. Math. Phys.* **13** (1972), 1957–1984.
4. L. C. BIEDENHARN AND J. D. LOUCK, “On the structure of the canonical tensor operators in the unitary groups. II. The tensor operators in $U(3)$ characterized by the maximal null space,” *J. Math. Phys.* **13** (1972), 1985–2001.
5. L. C. BIEDENHARN AND J. D. LOUCK, “Angular Momentum in Quantum Physics: Theory and Applications,” Vol. 8 in *Encly. of Math. and Its Appl.* (G.-C. Rota, Ed.), Addison-Wesley, Reading, Mass., 1981.
6. L. C. BIEDENHARN AND J. D. LOUCK, “The Racah-Wigner Algebra in Quantum Theory,” Vol. 9 in *Encly. of Math. and Its Appl.* (G.-C. Rota, Ed.), Addison-Wesley, Reading, Mass., 1981.
7. L. C. BIEDENHARN AND J. D. LOUCK, A pattern calculus for tensor operators in the unitary groups, *Comm. Math. Phys.* **8** (1968), 89–131.
8. L. C. BIEDENHARN, R. A. GUSTAFSON, M. A. LOHE, J. D. LOUCK, AND S. C. MILNE, Special functions and group theory in theoretical physics, in “Proceedings of Mathematisches Forschungsinstitut Oberwolfach,” March 13–19, 1983, Reidel, Dordrecht, 129–162.
9. L. C. BIEDENHARN, R. A. GUSTAFSON AND S. C. MILNE, $U(n)$ Wigner coefficients, the path sum formula and invariant G -functions, *Advan. in Appl. Math.* **6** (1985), 291–349.
10. M. CIFTAN AND L. C. BIEDENHARN, Combinatorial structure of state vectors in U_n . I. Hook patterns for maximal and semimaximal states in U_n , *J. Math. Phys.* **10** (1969), 221–232.
11. R. A. GUSTAFSON, A q -analog of the Racah-Wigner algebra, in preparation.
12. R. A. GUSTAFSON AND S. C. MILNE, Schur functions and the invariant polynomials characterizing $U(n)$ tensor operators, *Advan. in Appl. Math.* **4** (1983), 422–478.
13. R. A. GUSTAFSON AND S. C. MILNE, A new symmetry for Biedenharn’s G -functions and classical hypergeometric series, *Advan. in Math.* **57** (1985), 209–225.
14. I. G. MACDONALD, “Symmetric Functions and Hall Polynomial,” Oxford Univ. Press, England, 1979.
15. S. C. MILNE, Hypergeometric series well-poised in $SU(n)$ and a generalization of Biedenharn’s g -functions, *Advan. in Math.* **36** (1980), 169–211.
16. S. C. MILNE, A new symmetry related to $SU(n)$ for classical basic hypergeometric series, *Advan. in Math.* **57** (1985), 71–90.
17. S. C. MILNE, A q -analog of hypergeometric series well-poised in $SU(n)$ and invariant G -functions, *Advan. in Math.* **58** (1985), 1–60.
18. G. DE B. ROBINSON, “Representation Theory of the Symmetric Group,” Univ. of Toronto Press, Toronto, Canada, 1961.